ELSEVIER

Composite Structures 62 (2003) 27–39

COMPOSITE STRUCTURES

www.elsevier.com/locate/compstruct

Linear static analysis and finite element modeling for laminated composite plates using third order shear deformation theory

M. Rastgaar Aagaah, M. Mahinfalah *, G. Nakhaie Jazar

Department of Mechanical Engineering and Applied Mechanics, North Dakota State University, Fargo, ND 58105, USA

Abstract

In this paper, deformations of a laminated composite plate due to mechanical loads are presented. Third order shear deformation theory of plates, which is categorized in equivalent single layer theories, is used to derive linear dynamic equations of a rectangular multi-layered composite plate. Moreover, derivation of equations for FEM and numerical solutions for displacements and stress distributions of different points of the plate with a sinusoidal distributed mechanical load for Navier type boundary conditions are presented.

Keywords: Third order shear deformation; Laminated composite plate; Finite element

1. Introduction

The use of composite materials in structural components are increasing due to their attractive properties such as high strength-to-weight ratio, ability to tailor the structural properties, etc. Plate structures find numerous applications in the aerospace, military and automotive industries. The effects of transverse shear deformation are considerable for composite structures, because of their high ratio of extensional modulus to transverse shear modulus.

Most of the structural theories used till now to characterize the behavior of composite laminates fall into the category of equivalent single layer (ESL) theories. In these theories, the material properties of the constituent layers are combined to form a hypothetical single layer whose properties are equivalent to through-the-thickness integrated sum of its constituents. This category of theories has been found to be adequate in predicting global response characteristics of laminates, like maximum deflections, maximum stresses, fundamental frequencies, or critical buckling loads [1].

Third order shear deformation theory, which is one of the ESL theories, is derived. This theory is based on the same assumptions as the classical (CLPT) and first order shear deformation plate theories (FSDT), except that the assumption on the straightness and normality of the transverse normal is relaxed [2–4].

Theories higher than third order are not used because the accuracy gained is so little that the effort required to solve the equations is not justified [5]. In single layer displacement-based theories, one single expansion for each displacement component is used through the entire thickness, and therefore, the transverse strains are continuous through the thickness, a strain state appropriate for homogeneous plates [5–7].

In the present work, the equations of motion have been derived for the linear deformation of laminated plates subjected to a mechanical load based on a third order shear deformation plate theory in conjunction with the Von Karman strains. Unlike to the first order shear deformation theory, the higher order theory does not require shear correction factors. Finally the finite element solution for the plate is derived.

2. Elasticity equations

The plate considered in this investigation consists of N orthotropic cross-ply and angle-ply layers with a total thickness h. Components of global Cartesian

* Corresponding author. Tel.: +1-701-231-8839; fax: +1-701-231-8913.

E-mail address: m.mahinfalah@ndsu.nodak.edu (M. Mahinfalah).

0263-8223/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0263-8223(03)00081-3

coordinates Ω, that is located at the middle of the plate, are (x, y, z) where x, y are in-plane coordinates, and z is the transverse coordinate. The top layer is at $z = -h/2$ and the bottom layer is located at $z = h/2$. Layer coordinates of a typical nth layer are Ω_n and its components are (x_n, y_n, z_n) and x_n is in the direction of fibers as shown in Fig. 1.

The linear constitutive equation of the nth layer when considering thermal expansion effect is given by

$$\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ & \overline{Q}_{22} & \overline{Q}_{23} & 0 & 0 & \overline{Q}_{26} \\ & & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ & & & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ & \text{Sym} & & & \overline{Q}_{55} & 0 \\ & & & & & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \bar{e}_1 - \bar{\alpha}_1\theta \\ \bar{e}_2 - \bar{\alpha}_2\theta \\ \bar{e}_3 - \bar{\alpha}_3\theta \\ \bar{e}_4 \\ \bar{e}_5 \\ \bar{e}_6 \end{Bmatrix} \tag{1}$$

where α_i are coefficients of thermal expansion in direction of layer coordinates and θ is the change in temperature of each layer. Since the thermal effects cause a volume change, they do not have effect on transverse stresses and strains. Therefore, thermal expansion coefficients for an orthotropic lamina have only three components [8,9]. Let θ be the angle between the layer coordinates and the global coordinate, then the following relationships exist between stresses and strains in both coordinates.

$$\begin{aligned} \{\bar{\sigma}\} &= [T]\{\sigma\} \\ \{\bar{e}\} &= [T]\{e\} \end{aligned} \tag{2}$$

σ and e are prescribed in global coordinate but $\bar{\sigma}$ and $\bar{e}$ are components of stress and strain in lamina coordinates. $[T]$ is rotational matrix about the transverse direction z at θ and is defined as

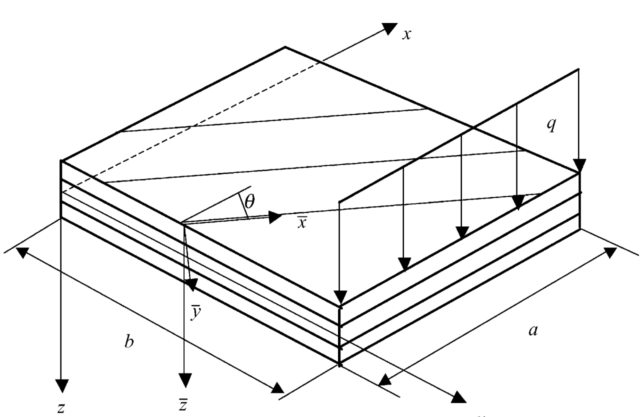

Fig. 1. Local and global coordinate systems of a laminate.

$$[T] = \begin{bmatrix} C^2 & S^2 & 0 & 0 & 0 & 2CS \\ S^2 & C^2 & 0 & 0 & 0 & -2CS \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & -S & 0 \\ 0 & 0 & 0 & S & C & 0 \\ -CS & CS & 0 & 0 & 0 & (C^2 - S^2) \end{bmatrix} \tag{3}$$

where $C = \cos(\theta)$ and $S = \sin(\theta)$.

From Eqs. (1) and (2), the following relation is held between elastic coefficients at two different coordinate systems.

$$[Q] = [T]^{-1}[\overline{Q}] \cdot [T] \tag{4}$$

$[\overline{Q}]$ and $[Q]$ are defined in terms of local coordinate of each layer and global coordinate of plate, respectively. After rotation of coordinates, in-plane thermal expansion coefficient α_6 will appear.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ & Q_{22} & Q_{23} & 0 & 0 & Q_{26} \\ & & Q_{33} & 0 & 0 & Q_{36} \\ & & & Q_{44} & Q_{45} & 0 \\ & \text{Sym} & & & Q_{55} & 0 \\ & & & & & Q_{66} \end{bmatrix} \begin{Bmatrix} e_1 - \alpha_1\theta \\ e_2 - \alpha_2\theta \\ e_3 - \alpha_3\theta \\ e_4 \\ e_5 \\ e_6 - \alpha_6\theta \end{Bmatrix} \tag{5}$$

The relationships of material properties at two different coordinate systems are presented in Appendix A.

The following displacement field that was introduced by Robbins and Reddy [5], is the displacement field of third order shear deformation plate theory (TRDT).

$$u = u_0 + z\phi_x - z^2\left(\frac{1}{2}\frac{\partial \phi z}{\partial x}\right) - z^3\left[C_1\left(\frac{\partial w_0}{\partial x} + \phi_x\right) + \frac{1}{3}\frac{\partial \varphi z}{\partial x}\right] \tag{6}$$

$$v = v_0 + z\phi_y - z^2\left(\frac{1}{2}\frac{\partial \phi z}{\partial y}\right) - z^3\left[C_1\left(\frac{\partial w_0}{\partial y} + \phi_y\right) + \frac{1}{3}\frac{\partial \varphi z}{\partial y}\right] \tag{7}$$

$$w = w_0 + z\phi_z + z^2\varphi_z \tag{8}$$

where

$$C_1 = \frac{4}{3h^2}, \quad u_0 = u(x, y, 0, t), \quad v_0 = v(x, y, 0, t) \quad \text{and} \quad w_0 = w(x, y, 0, t) \tag{9}$$

(u_0, v_0, w_0) are the displacements of transverse normal on plane $z = 0$ and (ϕ_x, ϕ_y) are rotations of transverse normal on plane $z = 0$. ϕ_z is extension of a transverse normal, and φ_z is interpreted as a higher order rotation of transverse normal.

The number of dependent variables in Eqs. (6)–(8) is only 7. The displacement field in Eqs. (6)–(8) accommodates quadratic variation of transverse shear strains (and hence stresses) and vanishing of transverse shear stresses on the top and bottom of a general laminate composed of monoclinic layers [10,5]. Thus there is no need to use shear correction factors in a third order theory. The third order theories provide a slight increase in accuracy relative to the first order shear deformation theory (FSDT) solution, at the expense of a significant increase in computational effort. Moreover, finite element models of third order theories that satisfy the vanishing of transverse shear stresses on the bounding planes have the disadvantage of requiring continuity of C^1 [5].

Using virtual work method,

$$\int_0^t (\delta U + \delta V - \delta K)\,\mathrm{d}t = 0 \tag{10}$$

equilibrium equations of the plate can be derived. U, V, and K are virtual strain energy, virtual work done by applied forces and virtual kinetic energy, respectively, and t is time.

Then the linear strains according to displacement field Eqs. (6)–(8) are

$$\begin{aligned} e_1 &= e_1^0 + z(k_1^0 + zk_1^1 + z^2k_1^2) \\ e_2 &= e_2^0 + z(k_2^0 + zk_2^1 + z^2k_2^2) \\ e_3 &= e_3^0 + z(k_3^0) \\ e_4 &= e_4^0 + z(k_4^1) = 2e_{23} \\ e_5 &= e_5^0 + z(k_5^1) = 2e_{13} \\ e_6 &= e_6^0 + z(k_6^0 + k_6^1 + k_6^2) \end{aligned} \tag{11}$$

It is assumed that there is an isothermal condition and temperature change and therefore thermal strains do not exist. In Appendix B, the relationships between strain components and displacement field Eqs. (6)–(8) are presented.

By substitution of stresses and strain and distributed force in Eq. (10), the final integral equation for plate elasticity is given by

$$\begin{aligned} \int_0^T \Bigg(&\int_{\Omega_0} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_1\delta e_1 + \sigma_2\delta e_2 + \sigma_3\delta e_3 + \sigma_4\delta e_4 \\ &+ \sigma_5\delta e_5 + \sigma_6\delta e_6)\,\mathrm{d}z\,\mathrm{d}A - \int_{\Omega_0} (q\delta w)\,\mathrm{d}A \\ &+ \int_{\Omega_0} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\ddot{u}\delta\ddot{u} + \ddot{v}\delta\ddot{v} + \ddot{w}\delta\ddot{w})\,\mathrm{d}z\,\mathrm{d}A \Bigg)\,\mathrm{d}t = 0 \end{aligned} \tag{12}$$

Plate inertias and stress resultants are defined as follow:

$$I_{1,\ldots,7} = \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} \rho_n(1, z, z^2, z^3, z^4, z^5, z^6)\,\mathrm{d}z \tag{13}$$

$$(N_i, M_i, S_i) = \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} \sigma_i(1, z, z^3)\,\mathrm{d}z \quad (i = 1, 2, 3, 6) \tag{14}$$

$$(Q_i) = \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} \sigma_i\,\mathrm{d}z \quad (i = 4, 5) \tag{15}$$

$$(P_i) = \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} \sigma_i z^2\,\mathrm{d}z \quad (i = 1, 2, 3, 4, 5, 6) \tag{16}$$

where $I_{1,\ldots,7}$ are inertias and N_i, M_i, S_i, Q_i and P_i are stress resultants. Using fundamental lemma of calculus of variation [10–12], the equation of motion of the plate can be written as

δu_0:

$$\begin{aligned} N_{1,x} + N_{6,y} = {} & I_1\ddot{u}_0 + I_2\ddot{\phi}_x - \frac{1}{2}I_3\frac{\partial\ddot{\phi}_z}{\partial x} - C_1I_4\frac{\partial\ddot{w}_0}{\partial x} \\ & - C_1I_4\ddot{\phi}_x - \frac{1}{3}I_4\frac{\partial\ddot{\varphi}_z}{\partial x} \end{aligned} \tag{17}$$

δv_0:

$$\begin{aligned} N_{2,y} + N_{6,x} = {} & I_1\ddot{v}_0 + I_2\ddot{\phi}_y - \frac{1}{2}I_3\frac{\partial\ddot{\phi}_z}{\partial y} - C_1I_4\frac{\partial\ddot{w}_0}{\partial y} \\ & - C_1I_4\ddot{\phi}_y - \frac{1}{3}I_4\frac{\partial\ddot{\varphi}_z}{\partial y} \end{aligned} \tag{18}$$

δw_0:

$$\begin{aligned} & C_1S_{1,xx} + C_1S_{2,yy} + Q_{4,y} + Q_{5,y} - 3C_1P_{4,y} - 3C_1P_{5,x} \\ & \quad + 2C_1S_{6,xy} + q \\ & \quad = I_1\ddot{w}_0 + I_2\ddot{\phi}_z + \frac{1}{3}\ddot{\varphi}_z + C_1I_4\frac{\partial\ddot{u}_0}{\partial x} + C_1I_4\frac{\partial\ddot{v}_0}{\partial y} \\ & \qquad + C_1I_5\frac{\partial\ddot{\phi}_x}{\partial x} + C_1I_5\frac{\partial\ddot{\phi}_y}{\partial y} - \frac{1}{2}C_1I_6\frac{\partial^2\ddot{\phi}_z}{\partial x^2} - \frac{1}{2}C_1I_6 \\ & \qquad \times \frac{\partial^2\ddot{\phi}_z}{\partial y^2} - C_1^2I_7\frac{\partial^2\ddot{w}_0}{\partial x^2} - C_1^2I_7\frac{\partial^2\ddot{w}_0}{\partial y^2} - C_1^2I_7\frac{\partial\ddot{\phi}_x}{\partial x} \\ & \qquad - C_1^2I_7\frac{\partial\ddot{\phi}_y}{\partial y} - \frac{1}{3}I_7\frac{\partial^2\ddot{\varphi}_z}{\partial x^2} - \frac{1}{3}I_7\frac{\partial^2\ddot{\varphi}_z}{\partial y^2} \end{aligned} \tag{19}$$

$\delta\phi_x$:

$$\begin{aligned} & M_{1,x} - C_1S_{1,x} - Q_5 + Q_{5,y} + 3C_1P_5 + M_{6,y} - C_1S_{6,y} \\ & \quad = (I_2 - C_1I_4)\ddot{u}_0 + (I_3 - 2C_1I_5 + C_1^2I_7)\ddot{\phi}_x + \frac{1}{2}(I_2 - C_1I_4) \\ & \qquad \times \frac{\partial\ddot{\phi}_z}{\partial x} + (-C_1I_5 + C_1^2I_7)\frac{\partial\ddot{w}_0}{\partial x} + \frac{1}{3}(-I_5 + C_1I_7)\frac{\partial\ddot{\varphi}_z}{\partial x} \end{aligned} \tag{20}$$

$\delta\phi_y$:

$$\begin{aligned} & M_{2,x} - C_1S_{2,x} - Q_4 + Q_{5,y} + 3C_1P_4 + M_{6,x} - C_1S_{6,x} \\ & \quad = (I_2 - C_1I_4)\ddot{v}_0 + (I_3 - 2C_1I_5 + C_1^2I_7)\ddot{\phi}_y \\ & \qquad + \frac{1}{2}(-I_4 + C_1I_6)\frac{\partial\ddot{\phi}_z}{\partial y} + (-C_1I_5 + C_1^2I_7)\frac{\partial\ddot{w}_0}{\partial y} \\ & \qquad + \frac{1}{3}(-I_5 + C_1I_7)\frac{\partial\ddot{\varphi}_z}{\partial y} \end{aligned} \tag{21}$$

$\delta\phi_z$:

$$\frac{1}{2}P_{1,xx}+\frac{1}{2}P_{2,yy}+P_{6,xy}-N_3-q\frac{h}{2} = I_1\ddot{w}_0+I_2\ddot{\phi}_z+I_3\ddot{\varphi}_z+C_1I_4\frac{\partial\ddot{u}_0}{\partial x}+C_1I_4\frac{\partial\ddot{v}_0}{\partial y} + (C_1I_5-C_1^2I_7)\frac{\partial\ddot{\phi}_x}{\partial x}+(C_1I_5-C_1^2I_7)\frac{\partial\ddot{\phi}_y}{\partial y} - C_1^2I_7\frac{\partial^2\ddot{w}_0}{\partial x^2}-C_1^2I_7\frac{\partial^2\ddot{w}_0}{\partial y^2}-\frac{1}{2}C_1I_6\frac{\partial^2\ddot{\phi}_z}{\partial x^2} -\frac{1}{2}C_1I_6\frac{\partial^2\ddot{\phi}_z}{\partial y^2}-\frac{1}{3}I_7\frac{\partial^2\ddot{\varphi}_z}{\partial x^2}-\frac{1}{3}I_7\frac{\partial^2\ddot{\varphi}_z}{\partial y^2} \tag{22}$$

$\delta\phi_z$:

$$\frac{1}{3}S_{1,xx}+\frac{1}{3}P_{2,yy}+\frac{2}{3}P_{6,xy}-2M_3+q\frac{h^2}{4} = I_3\ddot{w}_0+I_4\ddot{\phi}_z+I_5\ddot{\varphi}_z+\frac{1}{3}I_4\frac{\partial\ddot{u}_0}{\partial x}+\frac{1}{3}I_4\frac{\partial\ddot{v}_0}{\partial y} + C_1I_4\frac{\partial\ddot{v}_0}{\partial y}+\frac{1}{3}(I_5-C_1I_7)\frac{\partial\ddot{\phi}_x}{\partial x}+\frac{1}{3}(I_5-C_1I_7)\frac{\partial\ddot{\phi}_y}{\partial y} -\frac{1}{3}C_1I_7\frac{\partial^2\ddot{w}_0}{\partial y^2}-\frac{1}{6}I_6\frac{\partial^2\ddot{\phi}_z}{\partial x^2}-\frac{1}{6}I_6\frac{\partial^2\ddot{\phi}_z}{\partial y^2}-\frac{1}{9}I_7\frac{\partial^2\ddot{\varphi}_z}{\partial x^2} -\frac{1}{9}I_7\frac{\partial^2\ddot{\varphi}_z}{\partial y^2} \tag{23}$$

where C_1 is defined as

$$C_1=\frac{4}{3h^2} \tag{24}$$

and q is distributed transverse load on the top surface.

3. Finite element modeling of equations

Using approximation equation for displacement field as

$$\begin{aligned} &u_i=\langle u_i\rangle\{N\} \quad \phi_{yi}=\langle\phi_{yi}\rangle\{N\}\\ &v_i=\langle v_i\rangle\{N\} \quad \phi_{zi}=\langle\phi_{zi}\rangle\{N\}\\ &w_i=\langle w_i\rangle\{N\} \quad \varphi_{zi}=\langle\varphi_{zi}\rangle\{N\}\\ &\phi_{xi}=\langle\phi_{xi}\rangle\{N\} \end{aligned} \tag{25}$$

and substitution of displacements approximations in Eqs. (13)–(16) and (17)–(23), finite element type of elasticity equations can be derived. For writing the equations in displacement field parameters, following relations have to be used.

$$\left\{\begin{matrix}\left\{\begin{matrix}N_1\\N_2\\N_3\\N_6\end{matrix}\right\}\\\left\{\begin{matrix}M_1\\M_2\\M_3\\M_6\end{matrix}\right\}\\\left\{\begin{matrix}P_1\\P_2\\P_3\\P_6\end{matrix}\right\}\\\left\{\begin{matrix}S_1\\S_2\\S_3\\S_6\end{matrix}\right\}\end{matrix}\right\} = \begin{bmatrix}[A] & [B] & [D] & [E]\\ & [D] & [E] & [F]\\ & & [F] & [H]\\ \text{Sym} & & & [J]\end{bmatrix} \left\{\begin{matrix}\left\{\begin{matrix}e_1^0\\e_2^0\\e_3^0\\e_6^0\end{matrix}\right\}\\\left\{\begin{matrix}k_1^0\\k_2^0\\k_3^0\\k_6^0\end{matrix}\right\}\\\left\{\begin{matrix}k_1^1\\k_2^1\\k_3^1\\k_6^1\end{matrix}\right\}\\\left\{\begin{matrix}k_1^2\\k_2^2\\k_3^2\\k_6^2\end{matrix}\right\}\end{matrix}\right\} \tag{26}$$

Eq. (26) provides the relations between stress resultants and strains, which are defined by displacement field parameters. In the Eq. (26), the elements of stiffness matrix can for example be defined as

$$[A]=\begin{bmatrix}A_{11} & A_{12} & A_{13} & A_{16}\\ & A_{22} & A_{23} & A_{26}\\ & & A_{33} & A_{36}\\ \text{Sym} & & & A_{66}\end{bmatrix} \tag{27}$$

Elements of matrices $[A]$, $[B]$, $[D]$, $[E]$, $[F]$ and $[H]$ are defined as follows:

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij},J_{ij})=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{ij}(1,z,z^2,z^3,z^4,z^5,z^6,z^7)\,\mathrm{d}z \quad (i,j=1,2,3,6) \tag{28}$$

For in-plane components, the matrix form of stress resultants and strains are

$$\left\{\begin{matrix}\left\{\begin{matrix}Q_4\\Q_5\end{matrix}\right\}\\\left\{\begin{matrix}P_4\\P_5\end{matrix}\right\}\end{matrix}\right\} = \begin{bmatrix}[A] & [B] & [D] & [E]\\ [D] & [E] & [F] & [H]\end{bmatrix} \left\{\begin{matrix}\left\{\begin{matrix}e_4^0\\e_5^0\end{matrix}\right\}\\\left\{\begin{matrix}k_4^0\\k_5^0\end{matrix}\right\}\end{matrix}\right\} \tag{29}$$

where for example

$$[A]=\begin{bmatrix}A_{44} & A_{45}\\ \text{Sym} & A_{55}\end{bmatrix} \tag{30}$$

and

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij})=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{ij}(1,z,z^2,z^3,z^4,z^5,z^6)\,\mathrm{d}z \quad (i,j=4,5) \tag{31}$$

Finally, using the finite element analysis equations of motion can be written in compact form as the following equation

$$[M]\{\ddot{X}\} + [K]\{X\} = \{F\} \tag{32}$$

Using quadratic 6 nodes triangular elements to satisfy C^1-continuity, and imposing the following boundary conditions for Navier type boundary conditions, governing equations can be solved. Elements of matrices $[K]$, $[M]$ and $\{F\}$ are presented in Appendixes C, D and E, respectively. The plate is simply supported at four edges therefore, primary boundary conditions are

$$\begin{aligned}
&u_0(x,0) = u_0(x,b) = 0\\
&\phi_x(x,0) = \phi_x(x,b) = 0\\
&v_0(0,y) = v_0(a,y) = 0\\
&\phi_y(0,y) = \phi_y(a,y) = 0\\
&w_0(x,0) = w_0(x,b) = w_0(0,y) = w_0(a,y) = 0
\end{aligned} \tag{33}$$

It is seen that the governing equations are in general dynamic form. To analyze the static behavior of the plate, the stiffness matrix $[K]$ and the force vector $\{F\}$ are needed.

4. Numerical solutions

Tables 1 and 2 contain nondimensionalized mid point deflections and stresses obtained with 3-D elasticity theory (ELS), TSDT, first order shear deformation theory (FSDT), and classical laminate plate theory (CLPT) for the following three problems:

Problem 1. A three-ply (0-90-0) square $(a/b = 1)$ laminate with equal thickness layers has been subjected to a sinusoidal distributed transverse load on top plane, and the results are presented in Table 1. The material properties of each ply is assumed as $E_1 = 175$ GPa, $E_2 = 7$ GPa, $G_{12} = G_{13} = 3.5$ GPa, $G_{23} = 1.4$ GPa and $v_{12} = v_{13} = 0.25$.

Problem 2. A four-ply (0-90-90-0) square $(a/b = 1)$ laminate with equal thickness layers has been subjected to a sinusoidal distributed transverse load on top plane and the results are presented in Table 2. The material properties are as Problem 1.

Problem 3. A comparison of maximum deflection using FSDT with correction factor of $k = 5/6$ and TRDT for an antisymmetric cross-ply by different number of layers and with the same boundary conditions and load distribution as previous examples. For this case, the material properties are $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $v_{12} = 0.25$ and $a/h = 10$. The results are shown in Table 3. FSDT and ELS results are presented in [10,13].

The following nondimensionalized quantities at specific points are presented in Tables and Graphs as a result of TSDT and are compared to CLPT, FSDT and ELS solutions of the problem [10,12].

$$\begin{aligned}
&\bar{w} = w_0\left(\frac{a}{2},\frac{b}{2}\right)\left(\frac{E_2h^3}{a^4q_0}\right) \quad \overline{\sigma}_{xx} = \sigma_{xx}\left(\frac{a}{2},\frac{b}{2},\frac{h}{2}\right)\left(\frac{h^2}{b^2q_0}\right)\\
&\overline{\sigma}_{yy} = \sigma_{yy}\left(\frac{a}{2},\frac{b}{2},\frac{h}{4}\right)\left(\frac{h^2}{b^2q_0}\right) \quad \overline{\sigma}_{xy} = \sigma_{xy}\left(0,0,\frac{h}{2}\right)\left(\frac{h^2}{b^2q_0}\right)\\
&\overline{\sigma}_{yz} = \sigma_{yz}\left(\frac{a}{2},0,0\right)\left(\frac{h}{bq_0}\right) \quad \overline{\sigma}_{xz} = \sigma_{xz}\left(0,\frac{b}{2},0\right)\left(\frac{h}{bq_0}\right)
\end{aligned} \tag{34}$$

5. Conclusions

Using the Reddy displacement field for third order shear deformation theory, a set of dynamic equations for modeling the behavior of a laminated plate is derived. Third order shears deformation theory (TRDT) of Reddy has 7 parameters in displacement field and

Table 1
Nondimensionalized maximum stresses and deflections at the mid point of a square simply supported (0-90-0) laminate

a/h	Method	$\overline{\sigma}_{xx}$	$\overline{\sigma}_{yz}$	$\bar{w}$
4	ELS	0.775	0.217	–
	TSDT	0.7392	0.1884	0.0197
	FSDT	0.4370	0.1561	0.0177
3-D ELS [13]			0.1968	
10	ELS	0.590	0.123	–
	TSDT	0.5713	0.1082	0.0773
	FSDT	0.5134	0.0915	0.0669
3-D ELS [13]			0.1108	
100	ELS	0.552	0.094	–
	TSDT	0.5426	0.0791	0.0463
	FSDT	0.5384	0.0703	0.0434
3-D ELS [13]			0.0827	
–	CLPT	0.5387	–	0.4313

Table 2
Nondimensionalized maximum stresses and deflections at the mid point of a square simply supported (0-90-90-0) laminate

a/h	Method	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{yz}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{xy}$	$\bar{w}$
4	ELS	0.720	0.663	0.292	0.219	0.0467	0.0195
	TSDT	0.681	0.647	0.244	0.211	0.0451	0.0190
	FSDT	0.406	0.576	0.196	0.140	0.0308	0.0170
3-D ELS [13]				0.280	0.269		
10	ELS	0.559	0.401	0.196	0.301	0.0275	0.00743
	TSDT	0.551	0.394	0.163	0.211	0.0451	0.00732
	FSDT	0.499	0.361	0.130	0.167	0.0241	0.00663
3-D ELS [13]				0.181	0.318		
100	ELS	0.539	0.276	0.141	0.337	0.0216	0.00437
	TSDT	0.539	0.275	0.129	0.308	0.0216	0.00435
	FSDT	0.538	0.270	0.101	0.178	0.0213	0.00435
3-D ELS [13]				0.139	0.337		
–	CLPT	0.539	0.270	0.139	0.337	0.0213	0.00432

Table 3
Nondimensionalized maximum deflection at a square simply supported antisymmetric cross-ply (0-90-...) laminate

a/h	$N = 2$		$N = 6$	
	FSDT	TSDT	FSDT	TSDT
4	0.021492	0.020102	0.015473	0.015423
10	0.012373	0.012253	0.006354	0.006375
20	0.011070	0.010241	0.005052	0.005055
50	0.010705	0.010701	0.004687	0.004687
100	0.010653	0.010653	0.004635	0.004635
CLPT	0.0106306		0.004618	

satisfies the vanishing of transverse shear stresses on the boundary planes. By deriving the dynamic equation of motion and equations in finite element form, stresses and transverse displacements of different points of plate are defined. From the results, it is clear that the third order theory gives more accurate results for deflections and stresses when compared to the first order shear deformation plate theory (FSDT) with correction factor for shear deformation of $k = 5/6$. It is known that the shear correction factor k depends on the lamina properties. The fact that no shear correction coefficients are needed in the third order theory makes it more convenient to use.

In Table 1, nondimensionalized stresses $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yz}$ are presented. It is seen that for span to thickness ratio $a/h = 4$, errors for TRDT results are 4.6% and 13.3%, respectively, but for FSDT these errors are 43.6% and 28.1%. For $a/h = 10$, these errors are reduced to 3.2% and 12.1% for TRDT and 13% and 26% for FSDT. For $a/h = 100$, these errors are 1.8% and 15.9% for TRDT and 2.5% and 25.5% for FSDT, respectively. It is seen that the TRDT results for $\bar{\sigma}_{yz}$ are very close to the stresses computed from three-dimensional elasticity of first order shear deformation theory for less amount of a/h. In Table 2, for a symmetric cross-ply, it is seen that the third order theory in comparison with the elasticity solution, predicts deflection by 2% while the first order theory predicts by about 12.8% for $a/h = 4$. The errors are 1.4% in TSDT and 10.8% in FSDT for $a/h = 10$. It is seen that the errors are reduced at higher a/h. For $a/h = 100$, the errors are 0.4% for both theories. Results for stresses are also presented in Table 2. The closer

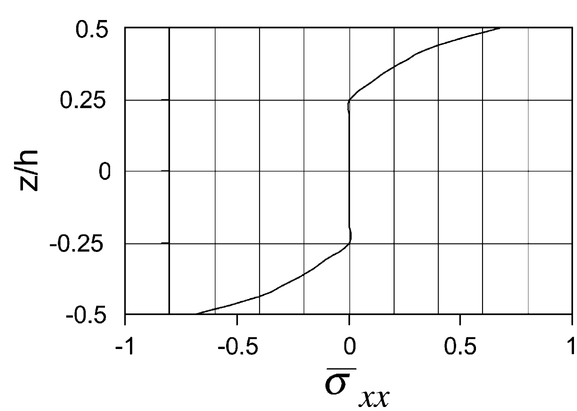

Fig. 2. Nondimensionalized normal stress $\bar{\sigma}_{xx}$ through the thickness of a 0-90-90-0 cross-ply with $a/h = 4$.

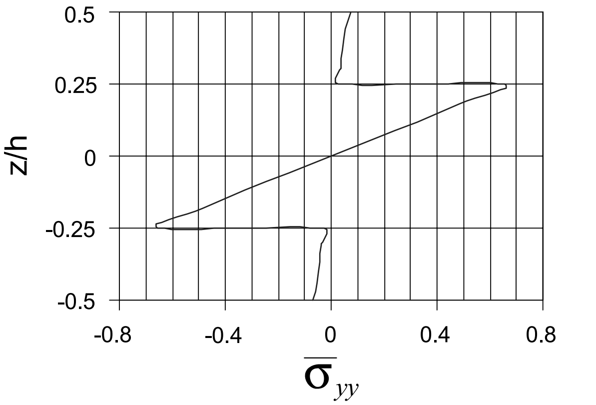

Fig. 3. Nondimensionalized normal stress $\overline{\sigma}_{yy}$ through the thickness of a 0-90-90-0 cross-ply with $a/h = 4$.

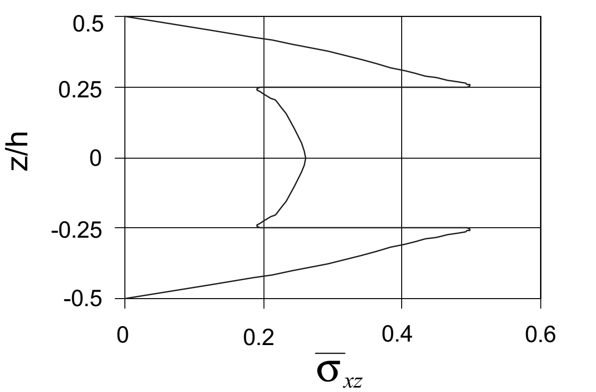

Fig. 4. Nondimensionalized transverse shear stress $\overline{\sigma}_{xz}$ through the thickness of a 0-90-90-0 cross-ply with $a/h = 4$.

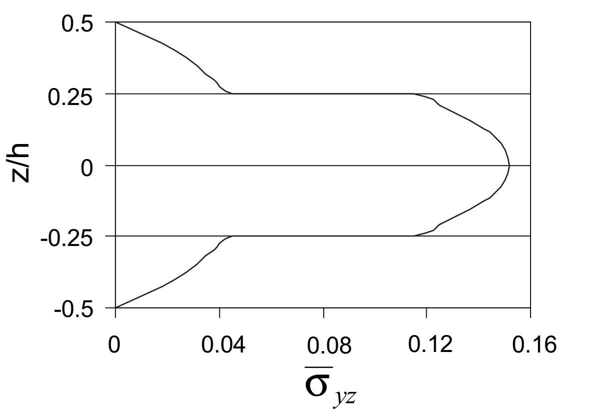

Fig. 5. Nondimensionalized transverse shear stress $\overline{\sigma}_{yz}$ through the thickness of a 0-90-90-0 cross-ply with $a/h = 4$.

results can be seen for in-plane stresses of TRDT and the results of elasticity solution. In Figs. 2–5 these through the thickness nondimensionalized stresses of the cross-ply with $a/h = 4$ are plotted. It is seen that the stresses are discontinuous like other ESL theories due to the continuity of the transverse shear strains through the thickness of lamina. In Table 3, deflection of an anti-symmetric cross-ply with 2 and 6 layers are presented. Relative errors for the results of TRDT and FSDT in 2-ply and 6-ply plates are 7.5% and 0.4% and for $a/h = 4$. These errors are near zero for $a/h = 100$.

Appendix A. Relationships of material properties in rotated coordinate systems

$$\alpha_1 = \overline{\alpha}_1 C^2 + \overline{\alpha}_2 S^2$$
$$\alpha_2 = \overline{\alpha}_1 S^2 + \overline{\alpha}_2 C^2$$
$$\alpha_3 = \overline{\alpha}_3$$
$$\alpha_4 = 0$$
$$\alpha_5 = 0$$
$$\alpha_6 = (\overline{\alpha}_1 - \overline{\alpha}_2) CS$$
$$Q_{11} = \overline{Q}_{11} C^4 + 2(\overline{Q}_{12} + 2\overline{Q}_{66}) C^2 S^2 + \overline{Q}_{22} S^4$$
$$Q_{12} = (\overline{Q}_{11} + \overline{Q}_{22} - 4\overline{Q}_{66}) C^2 S^2 + \overline{Q}_{12}(C^4 + S^4)$$
$$Q_{13} = \overline{Q}_{13} C^2 + \overline{Q}_{23} S^2$$
$$Q_{16} = -\overline{Q}_{22} CS^3 + \overline{Q}_{11} C^3 S - CS(C^2 - S^2)(\overline{Q}_{12} + 2\overline{Q}_{66})$$
$$Q_{22} = \overline{Q}_{11} S^4 + 2(\overline{Q}_{12} + 2\overline{Q}_{66}) C^2 S^2 + \overline{Q}_{22} C^4$$
$$Q_{23} = \overline{Q}_{13} S^2 + \overline{Q}_{23} C^2$$
$$Q_{26} = -\overline{Q}_{22} C^3 S + \overline{Q}_{11} CS^3 + CS(C^2 - S^2)(\overline{Q}_{12} + 2\overline{Q}_{66})$$
$$Q_{33} = \overline{Q}_{33}$$
$$Q_{44} = \overline{Q}_{44} C^2 + \overline{Q}_{55} S^2$$
$$Q_{45} = (\overline{Q}_{55} - \overline{Q}_{44}) CS$$
$$Q_{55} = \overline{Q}_{55} C^2 + \overline{Q}_{44} S^2$$
$$Q_{66} = (\overline{Q}_{11} + \overline{Q}_{22} - 2\overline{Q}_{12} - 2\overline{Q}_{66}) C^2 S^2 + \overline{Q}_{66}(C^4 + S^4)$$

Appendix B. Relationships between strain components and displacement

$$e_1^0 = \frac{\partial u_0}{\partial x}$$
$$k_1^0 = \frac{\partial \phi_x}{\partial x}$$
$$k_1^1 = -\frac{1}{2}\frac{\partial^2 \phi_z}{\partial x^2}$$
$$k_1^2 = -\left[C_1\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi_x}{\partial x}\right) + \frac{1}{3}\frac{\partial^2 w_z}{\partial x^2}\right]$$
$$e_2^0 = \frac{\partial v_0}{\partial y}$$
$$k_2^0 = \frac{\partial \phi_y}{\partial y}$$
$$k_2^1 = -\frac{1}{2}\frac{\partial^2 \phi_z}{\partial y^2}$$

$$k_2^2 = -\left[C_1\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \phi_y}{\partial y}\right) + \frac{1}{3}\frac{\partial^2 w_z}{\partial y^2}\right]$$

$$e_3^0 = \phi_z$$

$$k_3^0 = 2\varphi_z$$

$$k_3^1 = 0$$

$$k_3^2 = 0$$

$$e_4^0 = \frac{\partial w_0}{\partial y} + \phi_y$$

$$k_4^0 = 0$$

$$k_4^1 = -3C_1\left(\frac{\partial w_0}{\partial y} + \phi_y\right)$$

$$k_4^2 = 0$$

$$e_5^0 = \frac{\partial w_0}{\partial x} + \phi_x$$

$$k_5^1 = -3C_1\left(\frac{\partial w_0}{\partial x} + \phi_x\right)$$

$$k_5^2 = 0$$

$$e_6^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$

$$k_6^0 = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}$$

$$k_6^1 = -\frac{\partial^2 u_0}{\partial x \partial y}$$

$$k_6^2 = -\left[C_1\left(2\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + \frac{2}{3}\frac{\partial^2 \varphi_z}{\partial x \partial y}\right]$$

Appendix C. Elements of local stiffness matrix

N is the column vector of shape function and as an example for a quadratic triangular element, multiplication of two shape function is written as follow:

$$NN = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{Bmatrix} \langle N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \rangle$$

also for simplicity, the following terms are used.

$$N_x = \frac{\partial N}{\partial x} \quad N_y = \frac{\partial N}{\partial y} \quad N_{xx} = \frac{\partial^2 N}{\partial x^2}$$

$$N_{yy} = \frac{\partial^2 N}{\partial y^2} \quad N_{xy} = \frac{\partial^2 N}{\partial x \partial y}$$

$$K_{1,1} = \int_A (A_{11}N_xN_x + A_{16}N_xN_y + A_{61}N_yN_x + A_{66}N_yN_y)\,\mathrm{d}A$$

$$K_{1,2} = \int_A (A_{12}N_xN_y + A_{16}N_xN_x + A_{62}N_yN_y + A_{66}N_yN_x)\,\mathrm{d}A$$

$$K_{1,3} = \int_A (-C_1E_{11}N_xN_{xx} - C_1E_{12}N_xN_{yy} - 2C_1E_{16}N_xN_{xy} - C_1E_{61}N_yN_{xx} - C_1E_{62}N_yN_{yy} - 2C_1E_{66}N_yN_{xy})\,\mathrm{d}A$$

$$K_{1,4} = \int_A (B_{11}N_xN_x + B_{16}N_xN_y - C_1E_{11}N_xN_x - C_1E_{16}N_xN_y + B_{61}N_yN_x + B_{66}N_yN_y - C_1E_{61}N_yN_x - C_1E_{66}N_yN_y)\,\mathrm{d}A$$

$$K_{1,5} = \int_A (B_{12}N_xN_y + B_{16}N_xN_x - C_1E_{12}N_xN_y - C_1E_{16}N_xN_x + E_{62}N_yN_y + B_{66}N_yN_x - C_1E_{62}N_yN_y - C_1E_{66}N_yN_x)\,\mathrm{d}A$$

$$K_{1,6} = \int_A \left(A_{13}N_xN - \frac{1}{2}D_{11}N_xN_{xx} - D_{16}N_xN_{xy} + A_{63}N_yN - \frac{1}{2}D_{61}N_yN_{xx} - D_{66}N_yN_{xy} - \frac{1}{2}D_{12}N_xN_{yy} - \frac{1}{2}D_{62}N_yN_{yy}\right)\mathrm{d}A$$

$$K_{1,7} = \int_A \left(2B_{13}N_xN - \frac{1}{3}E_{11}N_xN_{xx} - \frac{1}{3}E_{12}N_xN_{yy} - \frac{2}{3}E_{16}N_xN_{xy} + 2B_{63}N_yN - \frac{1}{3}E_{61}N_yN_{xx} - \frac{1}{3}E_{62}N_yN_{yy} - \frac{2}{3}E_{66}N_yN_{xy}\right)\mathrm{d}A$$

$$K_{2,1} = \int_A (A_{12}N_yN_x + A_{26}N_yN_y + A_{61}N_xN_x + A_{66}N_xN_y)\,\mathrm{d}A$$

$$K_{2,2} = \int_A (A_{22}N_yN_y + A_{26}N_yN_x + A_{62}N_xN_y + A_{66}N_xN_x)\,\mathrm{d}A$$

$$K_{2,3} = \int_A (-C_1E_{22}N_yN_{yy} - 2C_1E_{26}N_yN_{xy} - C_1E_{61}N_xN_{xx} - C_1E_{12}N_yN_{xx} - C_1E_{62}N_xN_{yy} - 2C_1E_{66}N_xN_{xy})\,\mathrm{d}A$$

$$K_{2,4} = \int_A (B_{12}N_yN_x + B_{26}N_yN_y - C_1E_{12}N_yN_x - C_1E_{26}N_yN_y + B_{61}N_xN_x + B_{66}N_xN_y - C_1E_{61}N_xN_x - C_1E_{66}N_xN_y)\,\mathrm{d}A$$

$$K_{2,5} = \int_A (B_{22}N_yN_y + B_{26}N_yN_x - C_1E_{22}N_yN_y - C_1E_{26}N_yN_x + B_{62}N_xN_y + B_{66}N_xN_x - C_1E_{62}N_xN_y - C_1E_{66}N_xN_x)\,\mathrm{d}A$$

$$K_{2,6} = \int_A \left(A_{23}N_yN - \frac{1}{2}D_{12}N_yN_{xx} - \frac{1}{2}D_{22}N_yN_{yy} - D_{26}N_yN_{xy} + A_{63}N_xN - \frac{1}{2}D_{61}N_xN_{xx} - \frac{1}{2}D_{62}N_xN_{yy} - D_{66}N_xN_{xy} \right) \mathrm{d}A$$

$$K_{2,7} = \int_A \left(2B_{23}N_yN - \frac{1}{3}E_{12}N_yN_{xx} - \frac{1}{3}E_{22}N_yN_{yy} - \frac{2}{3}E_{26}N_yN_{xy} + 2B_{63}N_xN - \frac{1}{3}E_{61}N_xN_{xx} - \frac{1}{3}E_{62}N_xN_{yy} - \frac{2}{3}E_{66}N_xN_{xy} \right) \mathrm{d}A$$

$$K_{3,1} = \int_A (-C_1E_{11}N_{xx}N_x - C_1E_{12}N_{yy}N_x - C_1E_{16}N_{xx}N_y - C_1E_{26}N_{yy}N_y - 2C_1E_{16}N_{xy}N_x - 2C_1E_{66}N_{xy}N_y)\,\mathrm{d}A$$

$$K_{3,2} = \int_A (-C_1E_{12}N_{xx}N_y - C_1E_{16}N_{xx}N_x - C_1E_{22}N_{yy}N_y - C_1E_{26}N_{yy}N_x - 2C_1E_{26}N_{xy}N_y - 2C_1E_{66}N_{xy}N_x)\,\mathrm{d}A$$

$$K_{3,3} = \int_A (C_1^2J_{11}N_{xx}N_{xx} + C_1^2J_{12}N_{xx}N_{yy} + C_1^2J_{12}N_{yy}N_{xx} + C_1^2J_{22}N_{yy}N_{yy} + 2C_1^2J_{16}N_{xy}N_{xx} + 2C_1^2J_{26}N_{xy}N_{yy} + 4C_1^2J_{66}N_{xy}N_{xy} + 2C_1^2J_{16}N_{xx}N_{xy} + A_{44}N_yN_y + A_{45}N_yN_x + A_{55}N_xN_x + A_{45}N_xN_y + 2C_1^2J_{26}N_{yy}N_{xy} - 6C_1D_{44}N_yN_y - 6C_1D_{45}N_yN_x - 6C_1D_{55}N_xN_x - 6C_1D_{45}N_xN_y + 9C_1^2F_{44}N_yN_y + 9C_1^2F_{45}N_yN_x + 9C_1^2F_{55}N_xN_x + 9C_1^2F_{45}N_xN_y)\,\mathrm{d}A$$

$$K_{3,4} = \int_A (-C_1F_{11}N_{xx}N_x - C_1F_{16}N_{xx}N_y + C_1^2J_{11}N_{xx}N_x + C_1^2J_{16}N_{xx}N_y - C_1F_{12}N_{yy}N_x - C_1F_{26}N_{yy}N_y + C_1^2J_{12}N_{yy}N_x + C_1^2J_{26}N_{yy}N_y - 2C_1F_{16}N_{xy}N_x - 2C_1F_{66}N_{xy}N_y + 2C_1^2J_{16}N_{xy}N_x + 2C_1^2J_{66}N_{xy}N_y + A_{45}N_yN - 6C_1D_{45}N_yN + A_{55}N_xN - 3C_1D_{45}N_xN - 3C_1D_{55}N_xN + 9C_1^2E_{45}N_yN + 9C_1^2E_{55}N_xN)\,\mathrm{d}A$$

$$K_{3,5} = \int_A (-C_1F_{12}N_{xx}N_y - C_1F_{16}N_{xx}N_x + C_1^2J_{12}N_{xx}N_y + C_1^2J_{16}N_{xx}N_x - C_1F_{22}N_{yy}N_y - C_1F_{26}N_{yy}N_x + C_1^2J_{22}N_{yy}N_y + C_1^2J_{26}N_{yy}N_x - 2C_1F_{26}N_{xy}N_y - 2C_1F_{66}N_{xy}N_x + 2C_1^2J_{26}N_{xy}N_y + 2C_1^2J_{66}N_{xy}N_x + A_{44}N_yN - 6C_1D_{44}N_yN + A_{45}N_xN - 3C_1D_{45}N_xN - 3C_1D_{45}N_xN + 9C_1^2F_{44}N_yN + 9C_1^2F_{45}N_xN)\,\mathrm{d}A$$

$$K_{3,6} = \int_A \left(-C_1E_{13}N_{xx}N + \frac{1}{2}C_1H_{11}N_{xx}N_{xx} + \frac{1}{2}C_1H_{12}N_{xx}N_{yy} + C_1H_{16}N_{xx}N_{xy} - C_1E_{23}N_{yy}N + \frac{1}{2}C_1H_{12}N_{yy}N_{xx} + \frac{1}{2}C_1H_{22}N_{yy}N_{yy} + C_1H_{26}N_{yy}N_{xy} - 2C_1E_{36}N_{xy}N + C_1H_{16}N_{xy}N_{xx} + C_1H_{26}N_{xy}N_{yy} + 2C_1H_{66}N_{xy}N_{xy} \right) \mathrm{d}A$$

$$K_{3,7} = \int_A \left(\frac{1}{3}C_1J_{11}N_{xx}N_{xx} + \frac{1}{3}C_1J_{12}N_{xx}N_{yy} + \frac{2}{3}C_1J_{16}N_{xx}N_{xy} - 2C_1F_{23}N_{yy}N + \frac{1}{3}C_1J_{12}N_{yy}N_{xx} + \frac{1}{3}C_1J_{22}N_{yy}N_{yy} + \frac{2}{3}C_1J_{26}N_{yy}N_{xy} - 2C_1F_{13}N_{xx}N - 4C_1F_{36}N_{xy}N + \frac{2}{3}C_1J_{16}N_{xy}N_{xx} + \frac{2}{3}C_1J_{26}N_{xy}N_{yy} + \frac{4}{3}C_1^2J_{66}N_{xy}N_{xy} \right) \mathrm{d}A$$

$$K_{4,1} = \int_A (B_{11}N_xN_x + B_{16}N_xN_y + B_{66}N_yN_y + B_{16}N_yN_x - C_1E_{11}N_xN_x - C_1E_{16}N_yN_x - C_1E_{66}N_yN_y - C_1E_{16}N_xN_y)\,\mathrm{d}A$$

$$K_{4,2} = \int_A (B_{12}N_xN_y + B_{16}N_xN_x + B_{26}N_yN_y + B_{66}N_yN_x - C_1E_{12}N_xN_y - C_1E_{16}N_xN_x - C_1E_{26}N_yN_y - C_1E_{66}N_yN_x)\,\mathrm{d}A$$

$$K_{4,3} = \int_A (-C_1F_{11}N_xN_{xx} - 2C_1F_{16}N_xN_{xy} - C_1F_{16}N_yN_{xx} - 2C_1F_{66}N_yN_{xy} + C_1^2J_{11}N_xN_{xx} + C_1^2J_{12}N_xN_{yy} + 2C_1^2J_{16}N_xN_{yy} + C_1^2J_{16}N_yN_{xx} - C_1F_{12}N_xN_{yy} - C_1F_{26}N_yN_{yy} + C_1^2J_{26}N_yN_{yy} + 2C_1^2J_{66}N_yN_{xy} + A_{45}NN_y + A_{55}NN_x - 6C_1D_{45}NN_y - 3C_1D_{45}NN_x - 3C_1D_{55}NN_x + 9C_1^2F_{45}NN_y + 9C_1^2F_{55}NN_x)\,\mathrm{d}A$$

$$K_{4,4} = \int_A (D_{11}N_xN_x + D_{16}N_xN_y - 2C_1F_{11}N_xN_x - 2C_1F_{16}N_xN_yD_{16}N_yN_x + D_{66}N_yN_y - 2C_1F_{16}N_yN_x - 2C_1F_{66}N_yN_y + C_1^2J_{11}N_xN_x + C_1^2J_{16}N_xN_y + C_1^2J_{16}N_yN_x + C_1^2J_{66}N_yN_y + A_{55}NN - 6C_1D_{55}NN + 9C_1^2F_{55}NN)\,\mathrm{d}A$$

$$K_{4,5} = \int_A (D_{12}N_xN_y + D_{16}N_xN_x - 2C_1F_{12}N_xN_y - 2C_1F_{16}N_xN_xD_{26}N_yN_y + D_{66}N_yN_x - 2C_1F_{26}N_yN_y - 2C_1F_{66}N_yN_x + C_1^2J_{12}N_xN_y + C_1^2J_{16}N_xN_x + C_1^2J_{26}N_yN_y + C_1^2J_{66}N_yN_x + A_{45}NN - 6C_1D_{45}NN + 9C_1^2F_{45}NN)\,\mathrm{d}A$$

$$K_{4,6} = \int_A \left(B_{13}N_xN - \frac{1}{2}E_{11}N_xN_{xx} - \frac{1}{2}E_{12}N_xN_{yy} - E_{16}N_xN_{xy} + B_{36}N_yN - \frac{1}{2}E_{16}N_yN_{xx} - \frac{1}{2}E_{26}N_yN_{yy} - E_{66}N_yN_{xy} - C_1E_{13}N_xN + \frac{1}{2}C_1H_{11}N_xN_{xx} + \frac{1}{2}C_1H_{12}N_xN_{yy} + C_1H_{16}N_xN_{xy} - C_1E_{36}N_yN + \frac{1}{2}C_1H_{16}N_yN_{xx} + \frac{1}{2}C_1H_{26}N_yN_{yy} + C_1H_{66}N_yN_{xy}\right)\mathrm{d}A$$

$$K_{4,7} = \int_A \left(2D_{13}N_xN - \frac{1}{3}F_{11}N_xN_{xx} - \frac{1}{3}F_{12}N_xN_{yy} - \frac{2}{3}F_{16}N_xN_{xy} + 2D_{36}N_yN - \frac{1}{3}F_{16}N_yN_{xx} - \frac{1}{3}F_{26}N_yN_{yy} - \frac{2}{3}F_{66}N_yN_{xy} - 2C_1F_{13}N_xN + \frac{1}{3}C_1J_{11}N_xN_{xx} + \frac{1}{3}C_1J_{12}N_xN_{yy} + \frac{2}{3}C_1J_{16}N_xN_{xy} - 2C_1F_{36}N_yN + \frac{1}{3}C_1J_{16}N_yN_{xx} + \frac{1}{3}C_1J_{26}N_yN_{yy} + \frac{2}{3}C_1J_{66}N_yN_{xy}\right)\mathrm{d}A$$

$$K_{5,1} = \int_A (B_{12}N_yN_x + B_{16}N_xN_x + B_{26}N_yN_y + B_{66}N_xN_y - C_1E_{12}N_yN_x - C_1E_{16}N_xN_x - C_1E_{26}N_yN_y - C_1E_{66}N_xN_y)\,\mathrm{d}A$$

$$K_{5,2} = \int_A (B_{22}N_yN_y + B_{26}N_xN_y + B_{26}N_yN_x + B_{66}N_xN_x - C_1E_{22}N_yN_y - C_1E_{26}N_xN_y - C_1E_{26}N_yN_x - C_1E_{66}N_xN_x)\,\mathrm{d}A$$

$$K_{5,3} = \int_A (-C_1F_{12}N_yN_{xx} - C_1F_{22}N_yN_{yy} - 2C_1F_{26}N_yN_{xy} - C_1F_{16}N_xN_{xx} - C_1F_{26}N_xN_{yy} - 2C_1F_{66}N_xN_{xy} + C_1^2J_{12}N_yN_{xx} + C_1^2J_{22}N_yN_{yy} + 2C_1^2J_{26}N_yN_{xy} + C_1^2J_{16}N_xN_{xx} + C_1^2J_{26}N_xN_{yy} + 2C_1^2J_{66}N_xN_{xy} + A_{45}NN_x + A_{44}NN_y - 6C_1D_{44}NN_y - 6C_1D_{45}NN_x + 9C_1^2F_{44}NN_y + 9C_1^2F_{45}NN_x)\,\mathrm{d}A$$

$$K_{5,4} = \int_A (D_{12}N_yN_x + D_{26}N_yN_y - 2C_1F_{12}N_yN_x - 2C_1F_{26}N_yN_yD_{16}N_xN_x + D_{66}N_xN_y - 2C_1F_{16}N_xN_x - 2C_1F_{66}N_xN_y + C_1^2J_{12}N_yN_x + C_1^2J_{26}N_yN_y + C_1^2J_{16}N_xN_x + C_1^2J_{66}N_xN_y + A_{45}NN - 6C_1D_{45}NN + 9C_1^2F_{45}NN)\,\mathrm{d}A$$

$$K_{5,5} = \int_A (D_{22}N_yN_y + D_{26}N_yN_x - 2C_1F_{22}N_yN_y - 2C_1F_{26}N_yN_xD_{26}N_xN_y + D_{66}N_xN_x - 2C_1F_{26}N_xN_y - 2C_1F_{66}N_xN_x + C_1^2J_{22}N_yN_y + C_1^2J_{26}N_yN_x + C_1^2J_{26}N_xN_y + C_1^2J_{66}N_xN_x + A_{44}NN - 6C_1D_{44}NN + 9C_1^2F_{44}NN)\,\mathrm{d}A$$

$$K_{5,6} = \int_A \left(B_{23}N_yN - \frac{1}{2}E_{12}N_yN_{xx} - \frac{1}{2}E_{22}N_yN_{yy} - E_{26}N_yN_{xy} + B_{36}N_xN - \frac{1}{2}E_{16}N_xN_{xx} - \frac{1}{2}E_{26}N_xN_{yy} - E_{66}N_xN_{xy} - C_1E_{23}N_yN + \frac{1}{2}C_1H_{12}N_yN_{xx} + \frac{1}{2}C_1H_{22}N_yN_{yy} + C_1H_{26}N_yN_{xy} - C_1E_{36}N_xN + \frac{1}{2}C_1H_{16}N_xN_{xx} + \frac{1}{2}C_1H_{26}N_xN_{yy} + C_1H_{66}N_xN_{xy}\right)\mathrm{d}A$$

$$K_{5,7} = \int_A \left(2D_{23}N_yN - \frac{1}{3}F_{12}N_yN_{xx} - \frac{1}{3}F_{22}N_yN_{yy} - \frac{2}{3}F_{26}N_yN_{xy} + 2D_{36}N_xN - \frac{1}{3}F_{16}N_xN_{xx} - \frac{1}{3}F_{26}N_xN_{yy} - \frac{2}{3}F_{66}N_xN_{xy} - 2C_1F_{23}N_yN + \frac{1}{3}C_1J_{12}N_yN_{xx} + \frac{1}{3}C_1J_{22}N_yN_{yy} + \frac{2}{3}C_1J_{26}N_yN_{xy} - 2C_1F_{36}N_xN + \frac{1}{3}C_1J_{16}N_xN_{xx} + \frac{1}{3}C_1J_{26}N_xN_{yy} + \frac{2}{3}C_1J_{66}N_xN_{xy}\right)\mathrm{d}A$$

$$K_{6,1} = \int_A \left(A_{13}NN_x + A_{36}NN_y - D_{16}N_{xy}N_x - D_{66}N_{xy}N_y - \frac{1}{2}D_{11}N_{xx}N_x - \frac{1}{2}D_{16}N_{xx}N_y - \frac{1}{2}D_{12}N_{yy}N_x - \frac{1}{2}D_{26}N_{yy}N_y\right)\mathrm{d}A$$

$$K_{6,2} = \int_A \left(A_{23}NN_y + A_{36}NN_x - D_{26}N_{xy}N_y - D_{66}N_{xy}N_x - \frac{1}{2}D_{12}N_{xx}N_y - \frac{1}{2}D_{16}N_{xx}N_x - \frac{1}{2}D_{22}N_{yy}N_y - \frac{1}{2}D_{26}N_{yy}N_x\right)\mathrm{d}A$$

$$K_{6,3} = \int_A \Big(\frac{1}{2}C_1H_{11}N_{xx}N_{xx} + \frac{1}{2}C_1H_{12}N_{xx}N_{yy} + C_1H_{16}N_{xx}N_{xy} + \frac{1}{2}C_1H_{12}N_{yy}N_{xx} + \frac{1}{2}C_1H_{22}N_{yy}N_{yy} + C_1H_{26}N_{yy}N_{xy} + C_1H_{16}N_{xy}N_{xx} + C_1H_{26}N_{xy}N_{yy} + 2C_1H_{66}N_{xy}N_{xy} - C_1E_{13}NN_{xx} - C_1E_{23}NN_{yy} - 2C_1E_{36}NN_{xy} \Big)\, \mathrm{d}A$$

$$K_{6,4} = \int_A \Big(-\frac{1}{2}E_{11}N_{xx}N_x - \frac{1}{2}E_{16}N_{xx}N_y + \frac{1}{2}C_1H_{11}N_{xx}N_x + \frac{1}{2}C_1H_{16}N_{xx}N_y - \frac{1}{2}E_{12}N_{yy}N_x - \frac{1}{2}E_{26}N_{yy}N_y + \frac{1}{2}C_1H_{12}N_{yy}N_x + \frac{1}{2}C_1H_{26}N_{yy}N_y - E_{16}N_{xy}N_x - E_{66}N_{xy}N_y + C_1H_{16}N_{xy}N_x + C_1H_{66}N_{xy}N_y + B_{13}NN_x + B_{36}NN_y - C_1E_{13}NN_x - C_1E_{36}NN_y \Big)\, \mathrm{d}A$$

$$K_{6,5} = \int_A \Big(-\frac{1}{2}E_{12}N_{xx}N_y - \frac{1}{2}E_{16}N_{xx}N_x + \frac{1}{2}C_1H_{12}N_{xx}N_y + \frac{1}{2}C_1H_{16}N_{xx}N_x - \frac{1}{2}E_{22}N_{yy}N_y - \frac{1}{2}E_{26}N_{yy}N_x + \frac{1}{2}C_1H_{22}N_{yy}N_y + \frac{1}{2}C_1H_{26}N_{yy}N_x - E_{26}N_{xy}N_y - E_{66}N_{xy}N_x + C_1H_{26}N_{xy}N_y + C_1H_{66}N_{xy}N_x + B_{23}NN_y + B_{36}NN_x - C_1E_{23}NN_y - C_1E_{36}NN_x \Big)\, \mathrm{d}A$$

$$K_{6,6} = \int_A \Big(-\frac{1}{2}D_{13}N_{xx}N + \frac{1}{4}F_{12}N_{xx}N_{yy} + \frac{1}{4}F_{11}N_{xx}N_{xx} + \frac{1}{2}F_{16}N_{xx}N_{xy} - \frac{1}{2}D_{23}N_{yy}N + \frac{1}{4}F_{12}N_{yy}N_{xx} + \frac{1}{4}F_{22}N_{yy}N_{yy} + \frac{1}{2}F_{26}N_{yy}N_{xy} - D_{36}N_{xy}N + \frac{1}{2}F_{16}N_{xy}N_{xx} + \frac{1}{2}F_{26}N_{xy}N_{yy} + F_{66}N_{xy}N_{xy} + A_{33}NN - \frac{1}{2}D_{13}NN_{xx} - \frac{1}{2}D_{23}NN_{yy} - D_{36}NN_{xy} \Big)\, \mathrm{d}A$$

$$K_{6,7} = \int_A \Big(-E_{13}N_{xx}N + \frac{1}{6}H_{12}N_{xx}N_{yy} + \frac{1}{6}H_{11}N_{xx}N_{xx} + \frac{1}{3}H_{16}N_{xx}N_{xy} - E_{23}N_{yy}N + \frac{1}{6}H_{12}N_{yy}N_{xx} + \frac{1}{6}H_{22}N_{yy}N_{yy} + \frac{1}{3}H_{26}N_{yy}N_{xy} - 2E_{36}N_{xy}N + \frac{1}{3}H_{16}N_{xy}N_{xx} + \frac{1}{3}H_{26}N_{xy}N_{yy} + \frac{2}{3}H_{66}N_{xy}N_{xy} + 2B_{33}NN - \frac{1}{3}E_{13}NN_{xx} - \frac{1}{3}E_{23}NN_{yy} - \frac{2}{3}E_{36}NN_{xy} \Big)\, \mathrm{d}A$$

$$K_{7,1} = \int_A \Big(-\frac{1}{3}E_{11}N_{xx}N_x - \frac{1}{3}E_{16}N_{xx}N_y - \frac{1}{3}E_{12}N_{yy}N_x - \frac{1}{3}E_{26}N_{yy}N_y - \frac{2}{3}E_{16}N_{xy}N_x - \frac{2}{3}E_{66}N_{xy}N_y + 2B_{13}NN_x + 2B_{36}NN_y \Big)\, \mathrm{d}A$$

$$K_{7,2} = \int_A \Big(-\frac{1}{3}E_{12}N_{xx}N_y - \frac{1}{3}E_{16}N_{xx}N_x - \frac{1}{3}E_{22}N_{yy}N_y - \frac{1}{3}E_{26}N_{yy}N_x - \frac{2}{3}E_{26}N_{xy}N_y - \frac{2}{3}E_{66}N_{xy}N_x + 2B_{13}NN_y + 2B_{36}NN_x \Big)\, \mathrm{d}A$$

$$K_{7,3} = \int_A \Big(\frac{1}{3}C_1J_{11}N_{xx}N_{xx} + \frac{1}{3}C_1J_{12}N_{xx}N_{yy} + \frac{2}{3}C_1J_{16}N_{xx}N_{xy}\frac{1}{3}C_1J_{12}N_{yy}N_{xx} + \frac{1}{3}C_1J_{22}N_{yy}N_{yy} + \frac{2}{3}C_1J_{26}N_{yy}N_{xy} + \frac{2}{3}C_1J_{16}N_{xy}N_{xx} + \frac{2}{3}C_1J_{26}N_{xy}N_{yy} + \frac{4}{3}C_1J_{66}N_{xy}N_{xy} - 2C_1F_{13}NN_{xx} - 2C_1F_{23}NN_{yy} - 4C_1F_{36}NN_{xy} \Big)\, \mathrm{d}A$$

$$K_{7,4} = \int_A \Big(-\frac{1}{3}F_{11}N_{xx}N_x - \frac{1}{3}F_{16}N_{xx}N_y + \frac{1}{3}C_1J_{11}N_{xx}N_x + \frac{1}{3}C_1J_{16}N_{xx}N_y - \frac{1}{3}F_{12}N_{yy}N_x - \frac{1}{3}F_{26}N_{yy}N_y + \frac{1}{3}C_1J_{12}N_{yy}N_x + \frac{1}{3}C_1J_{26}N_{yy}N_y - \frac{2}{3}F_{16}N_{xy}N_x - \frac{2}{3}F_{66}N_{xy}N_y + \frac{2}{3}C_1J_{16}N_{xy}N_x + \frac{2}{3}C_1J_{66}N_{xy}N_y + 2D_{13}NN_x + 2D_{36}NN_y - 2C_1F_{13}NN_x - 2C_1F_{36}NN_y \Big)\, \mathrm{d}A$$

$$K_{7,5} = \int_A \Big(-\frac{1}{3}F_{12}N_{xx}N_y - \frac{1}{3}F_{16}N_{xx}N_x + \frac{1}{3}C_1J_{12}N_{xx}N_y + \frac{1}{3}C_1J_{16}N_{xx}N_x - \frac{1}{3}F_{22}N_{yy}N_y - \frac{1}{3}F_{26}N_{yy}N_x + \frac{1}{3}C_1J_{22}N_{yy}N_y + \frac{1}{3}C_1J_{26}N_{yy}N_x - \frac{2}{3}F_{26}N_{xy}N_y - \frac{2}{3}F_{66}N_{xy}N_x + \frac{2}{3}C_1J_{26}N_{xy}N_y + \frac{2}{3}C_1J_{66}N_{xy}N_x + 2D_{23}NN_y + 2D_{36}NN_x - 2C_1F_{23}NN_y - 2C_1F_{36}NN_x \Big)\, \mathrm{d}A$$

$$K_{7,6} = \int_A \bigg(-\frac{1}{3}E_{13}N_{xx}N + \frac{1}{6}H_{11}N_{xx}N_{xx} + \frac{1}{6}H_{12}N_{xx}N_{yy} + \frac{1}{3}H_{16}N_{xx}N_{xy} - \frac{1}{3}E_{23}N_{yy}N + \frac{1}{6}H_{12}N_{yy}N_{xx} + \frac{1}{6}H_{22}N_{yy}N_{yy} + \frac{1}{3}H_{26}N_{yy}N_{xy} - \frac{2}{3}E_{36}N_{xy}N + \frac{1}{3}H_{16}N_{xy}N_{xx} + \frac{1}{3}H_{26}N_{xy}N_{yy} + \frac{2}{3}H_{66}N_{xy}N_{xy} + 2B_{33}NN - E_{13}NN_{xx} - E_{23}NN_{yy} - 2E_{36}NN_{xy} \bigg) \mathrm{d}A$$

$$K_{7,7} = \int_A \bigg(-\frac{2}{3}F_{13}N_{xx}N + \frac{1}{9}J_{11}N_{xx}N_{xx} + \frac{1}{9}J_{12}N_{xx}N_{yy} + \frac{2}{9}J_{16}N_{xx}N_{xy} - \frac{2}{3}F_{23}N_{yy}N + \frac{1}{9}J_{12}N_{yy}N_{xx} + \frac{1}{9}J_{22}N_{yy}N_{yy} + \frac{2}{9}J_{26}N_{yy}N_{xy} - \frac{4}{3}F_{36}N_{xy}N + \frac{2}{9}J_{16}N_{xy}N_{xx} + \frac{2}{9}J_{26}N_{xy}N_{yy} + \frac{4}{9}J_{66}N_{xy}N_{xy} + 4D_{33}NN - \frac{2}{3}F_{13}NN_{xx} - \frac{2}{3}F_{23}NN_{yy} - \frac{4}{3}F_{36}NN_{xy} \bigg) \mathrm{d}A$$

Appendix D. Elements of local mass matrix

$$m_{1,1} = \int_A (I_1 NN)\,\mathrm{d}A$$
$$m_{1,2} = 0$$
$$m_{1,3} = \int_A (-C_1 I_4 NN_x)\,\mathrm{d}A$$
$$m_{1,4} = \int_A ((I_2 - C_1 I_4)NN)\,\mathrm{d}A$$
$$m_{1,5} = 0$$
$$m_{1,6} = \int_A \left(-\frac{1}{2} I_3 NN_x \right) \mathrm{d}A$$
$$m_{1,7} = \int_A \left(-\frac{1}{3} I_4 NN_x \right) \mathrm{d}A$$
$$m_{2,1} = 0$$
$$m_{2,2} = \int_A (I_1 NN)\,\mathrm{d}A$$
$$m_{2,3} = \int_A (-C_1 I_4 NN_y)\,\mathrm{d}A$$
$$m_{2,4} = 0$$
$$m_{2,5} = \int_A ((I_2 - C_1 I_4)NN)\,\mathrm{d}A$$
$$m_{2,6} = \int_A \left(-\frac{1}{2} I_3 NN_y \right) \mathrm{d}A$$
$$m_{2,7} = \int_A \left(-\frac{1}{3} I_4 NN_y \right) \mathrm{d}A$$
$$m_{3,1} = \int_A (-C_1 I_4 N_x N)\,\mathrm{d}A$$
$$m_{3,2} = \int_A (-C_1 I_4 N_y N)\,\mathrm{d}A$$
$$m_{3,3} = \int_A (C_1^2 I_7 N_x N_x + C_1^2 I_7 N_y N_y + I_1 NN)\,\mathrm{d}A$$
$$m_{3,4} = \int_A (-C_1(I_5 - C_1 I_7)N_x N)\,\mathrm{d}A$$
$$m_{3,5} = \int_A (-C_1(I_5 - C_1 I_7)N_y N)\,\mathrm{d}A$$
$$m_{3,6} = \int_A \left(\frac{1}{2} C_1 I_6 N_x N_x + \frac{1}{2} C_1 I_6 N_y N_y + I_2 NN \right) \mathrm{d}A$$
$$m_{3,7} = \int_A \left(\frac{1}{3} C_1 I_7 N_x N_x + \frac{1}{3} C_1 I_7 N_y N_y + I_3 NN \right) \mathrm{d}A$$
$$m_{4,1} = \int_A ((I_2 - C_1 I_4)NN)\,\mathrm{d}A$$
$$m_{4,2} = 0$$
$$m_{4,3} = \int_A (-C_1(I_5 - C_1 I_7)NN_x)\,\mathrm{d}A$$
$$m_{4,4} = \int_A ((I_3 - 2C_1 I_5 + C_1^2 I_7)NN)\,\mathrm{d}A$$
$$m_{4,5} = 0$$
$$m_{4,6} = \int_A \left(\frac{1}{2}(-I_4 + C_1 I_6)NN_x \right) \mathrm{d}A$$
$$m_{4,7} = \int_A \left(\frac{1}{3}(-I_5 + C_1 I_7)NN_x \right) \mathrm{d}A$$
$$m_{5,1} = 0$$
$$m_{5,2} = \int_A ((I_2 - C_1 I_4)NN)\,\mathrm{d}A$$
$$m_{5,3} = \int_A (-C_1(I_5 - C_1 I_7)NN_y)\,\mathrm{d}A$$
$$m_{5,4} = 0$$
$$m_{5,5} = \int_A ((I_3 - 2C_1 I_5 + C_1^2 I_7)NN)\,\mathrm{d}A$$
$$m_{5,6} = \int_A \left(\frac{1}{2}(-I_4 + C_1 I_6)NN_y \right) \mathrm{d}A$$
$$m_{5,7} = \int_A \left(\frac{1}{3}(-I_5 + C_1 I_7)NN_y \right) \mathrm{d}A$$
$$m_{6,1} = \int_A \left(-\frac{1}{2} I_3 N_x N \right) \mathrm{d}A$$
$$m_{6,2} = \int_A \left(-\frac{1}{2} I_3 N_y N \right) \mathrm{d}A$$
$$m_{6,3} = \int_A \left(\frac{1}{2} C_1 I_6 N_x N_x + \frac{1}{2} C_1 I_6 N_y N_y + I_2 NN \right) \mathrm{d}A$$

$$m_{6,4} = \int_A \left(-\frac{1}{2} I_4 N_x N + \frac{1}{2} C_1 I_6 N_x N \right) \mathrm{d}A$$

$$m_{6,5} = \int_A \left(-\frac{1}{2} I_4 N_y N + \frac{1}{2} C_1 I_6 N_y N \right) \mathrm{d}A$$

$$m_{6,6} = \int_A \left(\frac{1}{4} I_5 N_x N_x + \frac{1}{4} I_5 N_y N_y + I_3 NN \right) \mathrm{d}A$$

$$m_{6,7} = \int_A \left(\frac{1}{6} I_6 N_x N_x + \frac{1}{6} I_6 N_y N_y + I_4 NN \right) \mathrm{d}A$$

$$m_{7,1} = \int_A \left(-\frac{1}{3} I_4 N_x N \right) \mathrm{d}A$$

$$m_{7,2} = \int_A \left(-\frac{1}{3} I_4 N_y N \right) \mathrm{d}A$$

$$m_{7,3} = \int_A \left(\frac{1}{3} C_1 I_7 N_x N_x + \frac{1}{3} C_1 I_7 N_y N_y + I_3 NN \right) \mathrm{d}A$$

$$m_{7,4} = \int_A \left(-\frac{1}{3} I_5 N_x N + \frac{1}{3} C_1 I_7 N_x N \right) \mathrm{d}A$$

$$m_{7,5} = \int_A \left(-\frac{1}{3} I_5 N_y N + \frac{1}{3} C_1 I_7 N_y N \right) \mathrm{d}A$$

$$m_{7,6} = \int_A \left(\frac{1}{6} I_6 N_x N_x + \frac{1}{6} I_6 N_y N_y + I_4 NN \right) \mathrm{d}A$$

$$m_{7,7} = \int_A \left(\frac{1}{9} I_7 N_x N_x + \frac{1}{9} I_7 N_y N_y + I_5 NN \right) \mathrm{d}A$$

Appendix E. Elements of local force vector

$$f_1 = f_2 = 0$$

$$f_3 = \int_A (hqN)\, \mathrm{d}A$$

$$f_4 = f_5 = f_6 = 0$$

$$f_7 = \int_A \left(\frac{h^3}{4} qN \right) \mathrm{d}A$$

References

[1] Green AE, Naghdi PM. A theory of laminated composite plates. J Appl Math 1982;29(1):35–46.

[2] Bose P, Reddy JN. Analysis of composite plates using various plate theories. Part 1: Formulation and analytical solutions. Struct Eng Mech 1998;6(6):583–612.

[3] Liberescu L, Hause T. Recent developments in the modeling and behavior of advanced sandwich constructions: a survey. Compos Struct 2000;48:1–17.

[4] Lo KH, Christensen RM, Wu EM. A higher-order theory of isotropic elastic plates. J Appl Mech 1977;18:31–8.

[5] Robbins DH, Reddy JN. Structural theories and computational models for composite laminates. Appl Mech Rev 1994;47(6-1):147–70.

[6] Noor AK, Scott Burton W. Three-dimensional solutions for antisymmetrically laminated anisotropic plates. J Appl Mech, Trans ASME 1989;1:7.

[7] Noor AK, Scott W. Assessment of shear deformation theories for multilayered composite plates. Appl Mech Rev 1989;42(1):1–12.

[8] Whitney JM, Leissa AW. Analysis of heterogeneous anisotropic plates. J Appl Mech 1969;6:261–266.

[9] Whitney JM, Pagano NJ. Shear deformation in heterogeneous anisotropic elastic plates. J Appl Mech 1970;37:1031–6.

[10] Reddy JN. Mechanics of laminated composite plates. New York: CRC Press; 1997.

[11] Ochao OO, Reddy JN. Finite element analysis of composite laminate. Netherlands: Kluwer Academic Publishers; 1997.

[12] Reddy JN. Energy and Variational methods in applied mechanics. New York: John Wiley & Sons; 1984.

[13] Pagano NJ. Exact solutions for rectangular bi-directional composites and sandwich plates. AIAA J 1972;10:931–3.