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Natural frequencies of laminated composite plates using third order shear deformation theory

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Abstract

In this paper, natural frequencies of square laminated composite plates for different supports at edges are presented. Using a third order shear deformation theory of plates (TSDT), which is categorized in equivalent single layer theories (ESL), a new set of linear equations of motion for square multi-layered composite plates has been derived. Laminated plates are supposed to be either angle-ply or cross-ply. Moreover, FEM is used to solve the equations and find the fundamental natural frequencies. Finally some results for plates with different combination of layers and supports are reported. The results are compared to the results of other ESL.

Keywords: Natural frequency; Third order shear deformation (TSDT); Laminated composite plate; Finite element method (FEM)

1. Introduction

Laminated composite plates are widely used in industry and new fields of technology. Due to high degrees of anisotropy and low rigidity in transverse shear, Kirchhoff hypothesis as a classical theory is no longer adequate. The hypothesis states that transverse normal to the mid-plane of a plate remains straight and normal after deformation because of the negligible transverse shear effects. Refined theories based on removing those restrictions of transverse normal have been recently used. As a result, the free vibration frequencies calculated by using the classical thin plate theory are higher than those obtained by Mindlin plate theory in which transverse shear and rotary inertia effects are included [1].

Most of the structural theories used up to now, characterize the behavior of composite laminates fall into the category of equivalent single layer (ESL) theories. Different techniques for analysis of free vibration of composite laminated plates have been done. Reddy and Khedeir [2] presented analytical and finite element solutions for vibration and buckling of laminated composite plates using various plate theories to prove necessity of shear deformation theories to predict the behavior of composite laminates. Khedeir and Reddy [3] obtained a complete set of linear equations of the second order theory to analyze the free vibration behavior of cross-ply and antisymmetric angle-ply laminated plates. In these theories, the material properties of the constituent layers are combined to form a hypothetical single layer whose properties are equivalent to through-the-thickness integrated sum of its constituents. This category of theories has been found to be adequate in predicting global response characteristics of laminates like maximum deflections, maximum stresses, fundamental frequencies, or critical buckling loads [4].

We present a third order shear deformation formulation which is based on the same assumptions as the classical (CLPT) and first order shear deformation plate theories (FSDT), except that the assumption on the straightness and normality of the transverse normal is relaxed [4–6]. Theories higher than third order

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Nomenclature

C_1	parameter equal to $4/3h^2$	w	displacements of transverse normal in w direction
∂	differentiation operator	w_0	displacements of transverse normal on plane $z = 0$
e_i	strains in global coordinates	$\ddot{w}$	acceleration of transverse normal
e_i^0	strain components	x	in-plane global coordinate
E_i	modulus of elasticity in ith direction	x_n	in-plane local coordinate
$\{F\}$	force vector	$\{X\}$	displacement vector
h	plate thickness	$\{\ddot{X}\}$	acceleration vector
$I_{1\ldots7}$	plate inertia according to 1, z, z^2, z^3, z^4, z^5, z^6 respectively	y	in-plane global coordinate
$[K]$	stiffness matrix	y_n	in-plane local coordinate
k_i^j	strain components	z	out-of-plane global coordinate
$[M]$	mass matrix	z_n	out-of-plane local coordinate
M_i	stress resultants according to 1st order of z	δ	variational operator
N_i	stress resultants according to 0th order of z	ϕ_x, ϕ_y	rotations of transverse normal on plane $z = 0$
R_i	stress resultants according to 2nd order of z	$\ddot{\phi}_x$,$\ddot{\phi}_y$	acceleration of rotations of transverse normal on plane $z = 0$
q	distributed load	ϕ_z	extension of transverse normal
Q_{ij}	elastic coefficients in global coordinate	$\ddot{\phi}_z$	acceleration of extension of transverse normal
$\overline{Q}_{ij}$	elastic coefficients in local coordinate	φ_z	higher order rotation of transverse normal
Q_i	stress resultants according to 1st order of z	$\ddot{\varphi}_z$	acceleration of higher order rotation of transverse normal
P_i	stress resultants according to 3rd order of z	σ_i	ith component of stress in global coordinates
u	displacements of transverse normal in x direction	v_{ij}	Poisson's ratios
u_0	displacements of transverse normal on plane $z = 0$	θ	the angle between the layer coordinates and the global coordinate
$\ddot{u}$	acceleration of transverse normal	ω_m	natural frequencies
v	displacements of transverse normal in y direction	$\bar{\omega}$	nondimensional natural frequencies
v_0	displacements of transverse normal on plane $z = 0$	Ω	global Cartesian coordinates
$\ddot{v}$	acceleration of transverse normal	Ω_n	local Cartesian coordinates

are not used because the accuracy gained is so little that the effort required to solve the equations is not justified [7]. In single-layer displacement-based theories, one single expansion for each displacement component is used through the entire thickness, and therefore, the transverse strains are continuous through the thickness, a strain state appropriate for homogeneous plates [7–9].

In the present work, using shear deformation theory concept and a seven-parameter displacement field, a new set of equations of motion has been derived. Unlike the first order shear deformation theory, the higher order theory does not require shear correction factors. Equations of motion have been solved using finite elements method for a rectangular laminated composite plate. Different numbers of layers and different combinations for layers have been arranged. Both angle-ply and cross-ply laminates have been considered in this paper. Boundary conditions for plates are such that two of the edges, which are opposite, are always simply supported and the other two edges are used in different combinations of simply supported, free and clamped to be compared with the other reported studies. Comparisons between natural frequencies for different a/h as well as a variety of boundary conditions for several layers have been presented.

2. Equation of motion and mathematical treatments

The plate considered in this investigation consists of N orthotropic cross-ply and angle-ply layers with a total thickness h. Components of global Cartesian coordinates Ω, that is located at the middle of the plate, are (x, y, z) where x, y are in-plane coordinates, and z is the transverse coordinate. The top layer is at $z = h/2$ and the bottom layer is located at $z = -h/2$. Layer coordinates of a typical nth layer are Ω_n and its components are (x_n, y_n, z_n) and x_n is in the direction of fibers as shown in Fig. 1.

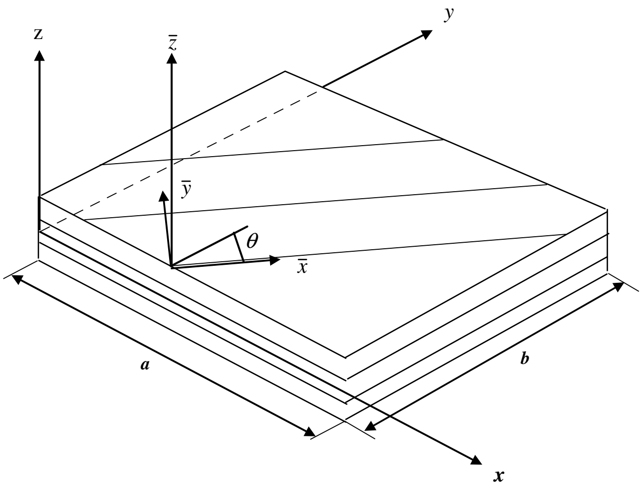

Fig. 1. Local and global coordinates system of a laminate.

The following displacement field is a third order displacement field and has seven independent variables u, v, w, ϕ_x, ϕ_y, ϕ_z and φ_z.

$$u = u_0 + z\phi_x - z^2\left(\frac{1}{2}\frac{\partial\phi_z}{\partial x}\right) - z^3\left[C_1\left(\frac{\partial w_0}{\partial x} + \phi_x\right) + \frac{1}{3}\frac{\partial\varphi_z}{\partial x}\right] \tag{1}$$

$$v = v_0 + z\phi_y - z^2\left(\frac{1}{2}\frac{\partial\phi_z}{\partial y}\right) - z^3\left[C_1\left(\frac{\partial w_0}{\partial y} + \phi_y\right) + \frac{1}{3}\frac{\partial\varphi_z}{\partial y}\right] \tag{2}$$

$$w = w_0 + z\phi_z + z^2\varphi_z \tag{3}$$

where

$$u_0 = u(x,y,0,t), \quad v_0 = v(x,y,0,t) \quad \text{and} \quad w_0 = w(x,y,0,t) \tag{4}$$

are the displacements of transverse normal on plane $z = 0$. ϕ_x and ϕ_y are rotations of transverse normal on plane $z = 0$, ϕ_z is extension of transverse normal, and φ_z is interpreted as a higher order rotation of transverse normal. According to displacement field Eqs. (1)–(3) the linear strains are

$$\begin{aligned} e_1 &= e_1^0 + z(k_1^0 + zk_1^1 + z^2k_1^2) \\ e_2 &= e_2^0 + z(k_2^0 + zk_2^1 + z^2k_2^2) \\ e_3 &= e_3^0 + z(k_3^0) \\ e_4 &= e_4^0 + z(k_4^1) = 2e_{23} \\ e_5 &= e_5^0 + z(k_5^1) = 2e_{13} \\ e_6 &= e_6^0 + z(k_6^0 + k_6^1 + k_6^2) \end{aligned} \tag{5}$$

In Appendix A, the relationships between strain components (5) and displacement field Eqs. (1)–(3) are presented.

The displacement field in Eqs. (1)–(3) accommodates quadratic variation of transverse shear strains (and hence stresses) and has to satisfy the condition of vanishing the transverse shear stresses on the top and bottom of a general laminate composed of monoclinic layers [7,10].

Writing transverse shear stresses

$$\sigma_4 = Q_{44}e_4 + Q_{45}e_5 = 0 \tag{6a}$$

$$\sigma_5 = Q_{45}e_4 + Q_{55}e_5 = 0 \tag{6b}$$

and substituting the transverse strains, we can write the following relation

$$\sigma_4 = \left(1 - \frac{3C_1h^2}{4}\right)\left(Q_{44}\left(\frac{\partial w_0}{\partial y} + \phi_y\right) + Q_{45}\left(\frac{\partial w_0}{\partial x} + \phi_x\right)\right) = 0 \tag{7a}$$

$$\sigma_5 = \left(1 - \frac{3C_1h^2}{4}\right)\left(Q_{45}\left(\frac{\partial w_0}{\partial y} + \phi_y\right) + Q_{55}\left(\frac{\partial w_0}{\partial x} + \phi_x\right)\right) = 0 \tag{7b}$$

It means that for satisfaction of the condition of zero transverse shear stresses on bonding plane, the following relation must exist.

$$\left(1 - \frac{3C_1h^2}{4}\right) = 0 \quad \text{or} \quad C_1 = \frac{4}{3h^2} \tag{8}$$

Thus, there is no need to use shear correction factors in a third-order theory. This third-order theory provides a slight increase in accuracy relative to the first order shear deformation theory (FSDT) solution at the expense of a significant increase in computational effort. Moreover, finite element models of third order theories that satisfy the vanishing of transverse shear stresses on the bounding planes require continuity of C^1 [7].

Equations of motion of the plate can be derived using virtual work method. For this reason, stress-strain relations in different coordinates and relationships for material properties in rotated coordinate systems have been used [11].

The equation of motion of the plate can be written as:

δu_0:

$$N_{1,x} + N_{6,y} = I_1\ddot{u}_0 + I_2\ddot{\phi}_x - \frac{1}{2}I_3\frac{\partial\ddot{\phi}_z}{\partial x} - C_1I_4\frac{\partial\ddot{w}_0}{\partial x} - C_1I_4\ddot{\phi}_x - \frac{1}{3}I_4\frac{\partial\ddot{\varphi}_z}{\partial x} \tag{9}$$

δv_0:

$$N_{2,y} + N_{6,x} = I_1\ddot{v}_0 + I_2\ddot{\phi}_y - \frac{1}{2}I_3\frac{\partial\ddot{\phi}_z}{\partial y} - C_1I_4\frac{\partial\ddot{w}_0}{\partial y} - C_1I_4\ddot{\phi}_y - \frac{1}{3}I_4\frac{\partial\ddot{\varphi}_z}{\partial y} \tag{10}$$

δw_0:

$$
\begin{aligned}
&C_1P_{1,xx} + C_1P_{2,yy} + Q_{4,y} + Q_{5,y} - 3C_1R_{4,y} \\
&\quad - 3C_1R_{5,x} + 2C_1P_{6,xy} + q \\
&\quad = I_1\ddot{w}_0 + I_2\ddot{\phi}_z + \frac{1}{3}\ddot{\varphi}_z + C_1I_4\frac{\partial \ddot{u}_0}{\partial x} + C_1I_4\frac{\partial \ddot{v}_0}{\partial y} \\
&\qquad + C_1I_5\frac{\partial \ddot{\phi}_x}{\partial x} + C_1I_5\frac{\partial \ddot{\phi}_y}{\partial y} - \frac{1}{2}C_1I_6\frac{\partial^2 \ddot{\phi}_z}{\partial x^2} - \frac{1}{2}C_1I_6\frac{\partial^2 \ddot{\phi}_z}{\partial y^2} \\
&\qquad - C_1^2I_7\frac{\partial^2 \ddot{w}_0}{\partial x^2} - C_1^2I_7\frac{\partial^2 \ddot{w}_0}{\partial y^2} - C_1^2I_7\frac{\partial \ddot{\phi}_x}{\partial x} - C_1^2I_7\frac{\partial \ddot{\phi}_y}{\partial y} \\
&\qquad - \frac{1}{3}I_7\frac{\partial^2 \ddot{\varphi}_z}{\partial x^2} - \frac{1}{3}I_7\frac{\partial^2 \ddot{\varphi}_z}{\partial y^2}
\end{aligned}
\tag{11}
$$

$\delta\phi_x$:

$$
\begin{aligned}
&M_{1,x} - C_1P_{1,x} - Q_5 + Q_{5,y} + 3C_1R_5 + M_{6,y} - C_1P_{6,y} \\
&\quad = (I_2 - C_1I_4)\ddot{u}_0 + (I_3 - 2C_1I_5 + C_1^2I_7)\ddot{\phi}_x \\
&\qquad + \frac{1}{2}(I_2 - C_1I_4)\frac{\partial \ddot{\phi}_z}{\partial x} + (-C_1I_5 + C_1^2I_7)\frac{\partial \ddot{w}_0}{\partial x} \\
&\qquad + \frac{1}{3}(-I_5 + C_1I_7)\frac{\partial \ddot{\varphi}_z}{\partial x}
\end{aligned}
\tag{12}
$$

$\delta\phi_y$:

$$
\begin{aligned}
&M_{2,x} - C_1P_{2,x} - Q_4 + Q_{5,y} + 3C_1R_4 + M_{6,x} - C_1P_{6,x} \\
&\quad = (I_2 - C_1I_4)\ddot{v}_0 + (I_3 - 2C_1I_5 + C_1^2I_7)\ddot{\phi}_y \\
&\qquad + \frac{1}{2}(-I_4 + C_1I_6)\frac{\partial \ddot{\phi}_z}{\partial y} + (-C_1I_5 + C_1^2I_7)\frac{\partial \ddot{w}_0}{\partial y} \\
&\qquad + \frac{1}{3}(-I_5 + C_1I_7)\frac{\partial \ddot{\varphi}_z}{\partial y}
\end{aligned}
\tag{13}
$$

$\delta\phi_z$:

$$
\begin{aligned}
&\frac{1}{2}R_{1,xx} + \frac{1}{2}R_{2,yy} + R_{6,xy} - N_3 - q\frac{h}{2} \\
&\quad = I_1\ddot{w}_0 + I_2\ddot{\phi}_z + I_3\ddot{\varphi}_z + C_1I_4\frac{\partial \ddot{u}_0}{\partial x} + C_1I_4\frac{\partial \ddot{v}_0}{\partial y} \\
&\qquad + (C_1I_5 - C_1^2I_7)\frac{\partial \ddot{\phi}_x}{\partial x} + (C_1I_5 - C_1^2I_7)\frac{\partial \ddot{\phi}_y}{\partial y} \\
&\qquad - C_1^2I_7\frac{\partial^2 \ddot{w}_0}{\partial x^2} - C_1^2I_7\frac{\partial^2 \ddot{w}_0}{\partial y^2} - \frac{1}{2}C_1I_6\frac{\partial^2 \ddot{\phi}_z}{\partial x^2} - \frac{1}{2}C_1I_6\frac{\partial^2 \ddot{\phi}_z}{\partial y^2} \\
&\qquad - \frac{1}{3}I_7\frac{\partial^2 \ddot{\varphi}_z}{\partial x^2} - \frac{1}{3}I_7\frac{\partial^2 \ddot{\varphi}_z}{\partial y^2}
\end{aligned}
\tag{14}
$$

$\delta\phi_z$:

$$
\begin{aligned}
&\frac{1}{3}P_{1,xx} + \frac{1}{3}R_{2,yy} + \frac{2}{3}R_{6,xy} - 2M_3 + q\frac{h^2}{4} \\
&\quad = I_3\ddot{w}_0 + I_4\ddot{\phi}_z + I_5\ddot{\varphi}_z + \frac{1}{3}I_4\frac{\partial \ddot{u}_0}{\partial x} + \frac{1}{3}I_4\frac{\partial \ddot{v}_0}{\partial y} + C_1I_4\frac{\partial \ddot{v}_0}{\partial y} \\
&\qquad + \frac{1}{3}(I_5 - C_1I_7)\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{1}{3}(I_5 - C_1I_7)\frac{\partial \ddot{\phi}_y}{\partial y} - \frac{1}{3}C_1I_7\frac{\partial^2 \ddot{w}_0}{\partial y^2} \\
&\qquad - \frac{1}{6}I_6\frac{\partial^2 \ddot{\phi}_z}{\partial x^2} - \frac{1}{6}I_6\frac{\partial^2 \ddot{\phi}_z}{\partial y^2} - \frac{1}{9}I_7\frac{\partial^2 \ddot{\varphi}_z}{\partial x^2} - \frac{1}{9}I_7\frac{\partial^2 \ddot{\varphi}_z}{\partial y^2}
\end{aligned}
\tag{15}
$$

where q is distributed transverse load on the top surface. Definitions of stress resultants and inertia terms are standard and reader can find them in reference texts such as [11].

Using approximation equation for following displacement field parameters

$$
\begin{aligned}
&u_i = \langle u_i\rangle\{N\} \quad \phi_{yi} = \langle \phi_{yi}\rangle\{N\} \\
&v_i = \langle v_i\rangle\{N\} \quad \phi_{zi} = \langle \phi_{zi}\rangle\{N\} \\
&w_i = \langle w_i\rangle\{N\} \quad \varphi_{zi} = \langle \varphi_{zi}\rangle\{N\} \\
&\phi_{xi} = \langle \phi_{xi}\rangle\{N\}
\end{aligned}
\tag{16}
$$

and substitution of displacements approximations in Eqs. (9)–(15), displacement based finite element model of elasticity equations can be derived and the equations can be set up in the following form

$$[M]\{\ddot{X}\} + [K]\{X\} = \{F\} \tag{17}$$

3. Natural frequencies for different boundary conditions

Using quadratic six nodes triangular elements to satisfy C^1-continuity of elements, and imposing the boundary conditions, governing equations can be solved to find fundamental frequencies. It is seen that the governing equations are in general dynamic form. To find the fundamental frequencies of the plate, the stiffness matrix $[K]$ and the mass matrix $[M]$ are needed. For linear problems the local stiffness matrix $[K]$ is independent of element displacement $\{X\}$ and the following relation is valid for all instants of time

$$\{\ddot{X}\} = -\omega^2\{X\} \tag{18}$$

Therefore, the global matrix equation can be written as

$$([K] - \omega_m^2[M])\{X\} = \{0\} \tag{19}$$

Natural frequencies of plate can be found from the above relation, where m is equal to $7 \times$ number of elements that are used.

We assume that the plate is simply supported at two edges while the boundary conditions at the other two edges, are a combination of boundary conditions. These combinations of supports are SSSS, SSSC, SCSC, SFSC, SFSS and SFSF, where S stands for simply support, C for clamped, and F for free boundary conditions. The first letter describes the boundary condition at $x = 0$, the second letter is the boundary condition at $y = 0$ and third and fourth letters describe the boundary conditions at $x = a$ and $y = b$ respectively. Primary boundary conditions that are used for displacement based finite element analysis are as follow:

For edges located at $x = 0$ and $x = a$ with simply support condition

$$\begin{aligned}
&v_0(0,y) = v_0(a,y) = 0\\
&w_0(0,y) = w_0(a,y) = 0\\
&\phi_y(0,y) = \phi_y(a,y) = 0\\
&\phi_z(0,y) = \phi_z(a,y) = 0\\
&\varphi_z(0,y) = \varphi_z(a,y) = 0\\
&N_1(0,y) = N_1(a,y) = 0\\
&M_1(0,y) = M_1(a,y) = 0\\
&P_1(0,y) = P_1(a,y) = 0\\
&S_1(0,y) = S_1(a,y) = 0
\end{aligned} \tag{20}$$

For the other two edges ($y = 0$ and $y = b$), boundary conditions are as follow:

For S (simply supported)

$$\begin{aligned}
&u_0 = w_0 = \phi_x = \phi_z = \varphi_z = 0\\
&N_2 = M_2 = P_2 = S_2 = 0
\end{aligned} \tag{21}$$

For C (clamped)

$$\begin{aligned}
&u_0 = v_0 = w_0 = \phi_x = \phi_y = \phi_z = \varphi_z = 0\\
&N_2 = N_6 = N_4 - C_1 N_4 = 0 \quad \text{and}\\
&M_2 = M_6 = S_2 = S_6 = P_2 = P_6 = 0
\end{aligned} \tag{22}$$

4. Numerical solutions

In Table 1, natural frequencies of different approaches that had been reported in other papers have been compared to the results of the present work. Results of present work (TSDT) for different ratio of longitudinal modulus to transverse modulus are presented. TSDT results show a small difference to the analytical Levy solution for other third order displacement fields. A significant improvement in the result in comparison to the FSDT and CLPT result is shown. Also it is seen that by increasing E_1/E_2, the difference between results of methods are decreasing.

Tables 2–9 present nondimensional natural frequencies of square laminated plates with different layers and boundary conditions. Plates are squared with sides ($a = b$).

Table 1
Nondimensional natural frequencies of a SSSS square laminated composite cross-ply (0/90/90/0) with different E_1/E_2 ratios and $a/h = 5$

E_1/E_2	TSDT[a]	HSDT [13][b]	FSDT [13]	CLPT [13]
10	8.2741	8.2940	8.2982	10.650
20	9.5312	9.5439	9.5671	13.948
30	10.2651	10.2840	10.3260	16.605
40	10.7912	10.7940	10.8540	18.891

[a] Present work, FEM of the equations governing in this paper.
[b] Results obtained by using other HSDT, FSDT and CLPT with Levy type solution.

Table 2
Nondimensional natural frequencies of a square angle-ply (45/−45/45/−45/45) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	11.196	19.059	21.371	24.302	25.949
SSSC	11.752	20.501	27.466	29.465	30.830
SCSC	11.595	19.333	26.303	29.581	30.018
SFSF	4.413	6.123	7.161	7.873	8.163
SFSS	6.953	9.593	11.790	12.480	13.001
SFSC	6.988	9.788	11.847	13.477	13.995

Table 3
Nondimensional natural frequencies of a square angle-ply (45/−45/45/−45) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	11.151	18.964	21.296	24.133	25.897
SSSC	11.705	20.399	27.370	29.260	30.768
SCSC	11.549	19.237	26.211	29.375	29.958
SFSF	4.395	6.093	7.136	7.818	8.147
SFSS	6.925	9.545	11.749	12.393	12.975
SFSC	6.960	9.739	11.806	13.383	13.967

Table 4
Nondimensional natural frequencies of a square angle-ply (45/−45/45) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	10.560	17.959	20.167	22.854	24.524
SSSC	11.085	19.318	25.919	27.709	29.137
SCSC	10.937	18.217	24.822	27.818	28.370
SFSF	4.162	5.770	6.758	7.404	7.715
SFSS	6.558	9.039	11.126	11.736	12.287
SFSC	6.591	9.223	11.180	12.674	13.227

Table 5
Nondimensional natural frequencies of a square angle-ply (45/−45) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	9.701	16.499	18.528	20.996	22.530
SSSC	10.183	17.747	23.812	25.456	26.768
SCSC	10.048	16.736	22.804	25.556	26.063
SFSF	3.824	5.301	6.208	6.802	7.088
SFSS	6.025	8.304	10.222	10.782	11.288
SFSC	6.055	8.473	10.271	11.643	12.151

Table 6
Nondimensional natural frequencies of a square cross-ply (0/90/0/90/0/90) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	9.016	12.300	14.130	15.040	15.177
SSSC	9.256	14.974	17.343	18.781	18.960
SCSC	10.287	17.522	23.448	30.392	34.179
SFSF	3.450	3.782	3.828	3.909	3.936
SFSS	3.818	4.283	4.406	4.461	4.617
SFSC	5.189	6.243	6.516	6.809	6.830

Table 7
Nondimensional natural frequencies of a square cross-ply (0/90/0/90) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	8.953	12.214	14.032	14.936	15.071
SSSC	9.191	14.870	17.222	18.650	18.829
SCSC	10.215	17.400	23.285	30.180	33.942
SFSF	3.426	3.755	3.802	3.882	3.909
SFSS	3.792	4.253	4.376	4.430	4.585
SFSC	5.153	6.199	6.471	6.761	6.783

Table 8
Nondimensional natural frequencies of a square cross-ply (0/90/0) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	8.935	12.19	14.004	14.906	15.041
SSSC	9.173	14.84	17.188	18.613	18.791
SCSC	10.195	17.365	23.239	30.12	33.874
SFSF	3.419	3.748	3.794	3.874	3.901
SFSS	3.784	4.245	4.367	4.421	4.576
SFSC	5.143	6.187	6.458	6.748	6.769

Table 9
Nondimensional natural frequencies of a square cross-ply (0/90) laminated composite plate with different support conditions

a/h	5	10	20	50	100
SSSS	8.220	11.215	12.884	13.714	13.838
SSSC	8.439	13.653	15.813	17.124	17.288
SCSC	9.379	15.976	21.380	27.710	31.164
SFSF	3.145	3.448	3.490	3.564	3.589
SFSS	3.481	3.905	4.018	4.067	4.210
SFSC	4.732	5.692	5.941	6.208	6.227

Natural frequencies of angle-plies are presented for a material with the following properties:

Material 1: $E_1 = 280$ GPa, $E_2 = 7$ GPa, $G_{12} = G_{13} = 4.2$ GPa, $G_{23} = 3.5$ GPa and $\nu_{12} = \nu_{13} = 0.25$.

Moreover, natural frequencies of cross-plies are presented for a material of the following properties:

Material 2: $E_1 = 175$ GPa, $E_2 = 7$ GPa, $G_{12} = G_{13} = 3.5$ GPa, $G_{23} = 1.4$ GPa and $\nu_{12} = \nu_{13} = 0.25$.

The following relation is used for presentation of nondimensional fundamental frequencies in this paper.

$$\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}} \tag{23}$$

Third order shear deformation theory (TRDT) of Reddy has seven parameters in displacement field and satisfies the vanishing of transverse shear stresses on the boundary planes. Using a third order displacement field, a set of dynamic equations for modeling the behavior of a laminated plate is derived. Applying displacement based finite element method to the governing equations, natural frequencies of laminated plates with different boundary conditions either cross-ply or angle-ply are calculated. Tables 2–5 present natural frequencies of angle-ply square plates. Table 2 is related to a (45/−45/45/−45/45) angle-ply. Table 3 is the natural frequencies of a (45/−45/45/−45) angle-ply, Table 4 shows the natural frequencies of a (45/−45/45) and Table 5 Shows the results for a (45/−45) angle-plies. The material properties for these plates are those of material 1. It is seen that growth of fundamental frequencies with respect to a/h are decreased when a/h is increased. Furthermore, with an increase in the number of layers, the rate of increase in natural frequencies will be reduced. The difference between natural frequencies of a two layer angle-ply and a three layer angle-ply is much more than the difference between a four layer and a five layer plate respectively. By considering the natural frequencies at $a/h = 10$, the relative difference between a three layer and a two layer angle-ply is 8.84%, between a four layer and a three layer is 5.59%, and between a five layer and a four layer is 0.5%. Tables 5–8 are natural frequencies of different cross-ply laminates. Table 6 is the results of a six layered (0/90/0/90/0/90) cross-ply. Tables 7–9 are natural frequencies of a (0/90/0/90), (0/90/0) and (0/90) cross-ply respectively. The material of the cross-ply laminates is material 2. The same pattern and a close range of relative difference can be seen in natural frequencies of plates when other support conditions presents. Same to an angle-ply plate; the rate of change in natural frequencies reduces by increasing a/h. Taking nondimensional fundamental frequencies at $a/h = 10$ shows a relative difference of 8.6% between a three layer and a two layer cross-plies. The relative difference is 1.9% between a four layer and a three layer and 0.7% between a six layer and a four layer respectively.

5. Conclusions

When comparing the same number of layers, angle-plies have higher natural frequencies than cross-plies. Increasing the number of layers causes a larger difference between their natural frequencies. It is seen that a cross-ply plate with SCSC boundary conditions, has a higher natural frequency than other types of boundary conditions, but for an angle-ply, the SSSC boundary conditions show a higher natural frequency than other boundary conditions. Also for a cross-ply, the natural frequencies of SFSF and SFSS boundary conditions are closer to each other and smaller than SFSC type. On the contrary, natural frequencies for an angle-ply, SFSS and SFSC boundary conditions are closer to each other and higher than the natural frequency of a plate with SFSF boundary conditions. Comparison between the natural frequencies of SSSS plates ($a/h = 5$) as a function of the orthotropicity ratio with the results of analytical levy solution for other displacement fields reported in [12,13] shows compatibility of the results of this new set of equations to the other shear deformation models.

Appendix A. Relationships between strain components and displacement

$$e_1^0 = \frac{\partial u_0}{\partial x} \qquad k_1^0 = \frac{\partial \phi_x}{\partial x}$$

$$k_1^1 = -\frac{1}{2}\frac{\partial^2 \phi_z}{\partial x^2} \qquad k_1^2 = -\left[C_1\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi_x}{\partial x}\right) + \frac{1}{3}\frac{\partial^2 w_z}{\partial x^2}\right]$$

$$e_2^0 = \frac{\partial v_0}{\partial y} \qquad k_2^0 = \frac{\partial \phi_y}{\partial y}$$

$$k_2^1 = -\frac{1}{2}\frac{\partial^2 \phi_z}{\partial y^2} \qquad k_2^2 = -\left[C_1\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \phi_y}{\partial y}\right) + \frac{1}{3}\frac{\partial^2 w_z}{\partial y^2}\right]$$

$$e_3^0 = \phi_z \qquad k_3^0 = 2\varphi_z$$

$$k_3^1 = 0 \qquad k_3^2 = 0$$

$$e_4^0 = \frac{\partial w_0}{\partial y} + \phi_y \qquad k_4^0 = 0$$

$$k_4^1 = -3C_1\left(\frac{\partial w_0}{\partial y} + \phi_y\right) \qquad k_4^2 = 0$$

$$e_5^0 = \frac{\partial w_0}{\partial x} + \phi_x \qquad k_5^0 = 0$$

$$k_5^1 = -3C_1\left(\frac{\partial w_0}{\partial x} + \phi_x\right) \qquad k_5^2 = 0$$

$$e_6^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \qquad k_6^0 = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}$$

$$k_6^1 = -\frac{\partial^2 u_0}{\partial x \partial y} \qquad k_6^2 = -\left[C_1\left(2\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + \frac{2}{3}\frac{\partial^2 \varphi_z}{\partial x \partial y}\right]$$

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