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TIME RESPONSE DYNAMICS OF LINEAR MODEL OF MICROCANTILEVER-MEMS

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Abstract

This paper presents analytic derivation of dynamic behavior of a liniearized micro-electro-mechanical resonator. The parametric oscillation results from a displacement-dependent electrostatic force generated by oscillation of a microbeam. The utilized device is a MEMS with a time-varying capacitor.

The stability and steady state dynamic behavior of the MEMS has been analyzed without polarization voltage. The main characteristic of the no-polarization model is effects of parameters in stability of the system. A set of stability charts is provided for prediction of the boundary between the stable and unstable domains for the principal resonance. Applying perturbation method, analytical equations are derived to describe both the steady state and time response of the system.

1. INTRODUCTION

A microelectromechanical resonator is built by attaching a microplate to the tip of a long microcantilever. The microplate is utilized as a moving electrode of a variable capacitor, whose other electrode is fixed to the frame of the MEMS device. The MEMS can be activated by both DC and AC voltages. The DC voltage is called polarization voltage and is utilized to activate the system statically. The polarization voltage generates an initial attraction due to electromagnetic force field between the electrodes, so increases the sensitivity of the system. However, introducing a polarization voltage destroys the symmetry of the system, and may be dropped in some applications. The AC voltage alternates the electromagnetic field and applies a harmonic force to the microplate. The alternative electromagnetic force excites the movable electrode harmonically. The mechanical stiffness of the microbeam or microcantilever struggles with the active force and produces a vibrating motion. An equivalent viscous damping may be used to simulate most damping phenomena in the system (Serry et. al. 1995).

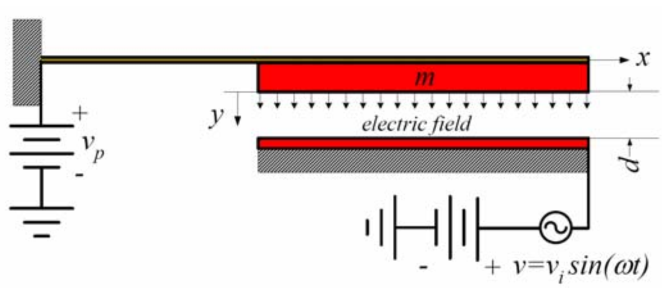

Figure 1. Illustration of a simplified time-varying mechanical model of the MEMS

The nonlinear electromagnetic force can be expanded into a Taylor series expansion. Then, the linear terms of the series may be substituted for the nonlinear electromagnetic force. The final equation of motion of the system is a parametric externally excited linear equation. Stability regions of a properly defined parameter space describe the stability behavior of the system and must be investigated. This paper focuses on the dynamic behavior of the nonpolarized and linearized MEMS, which is the simplest possible simulation, using perturbation methods. This model can be used as a starting point of simulation of MEMS dynamic, and is correct as long as the amplitude of oscillation is small enough. However, the system shows a very complicated stability diagram. A schematic model of the MEMS investigated in this study is illustrated in Figure 1.

2. MATHEMATICAL MODELING

Consider the mechanical model of the MEMS illustrated in Figure 1. The fixed electrode has an effective area *A*, and is connected to an alternating current voltage $v = v_i \sin(\omega t)$ where v_i, and ω, are the AC amplitude, and frequency, respectively. The other electrode is a plate of mass *m*, supported by a microbeam with stiffness *k*, and is not polarized. The system is also assumed to have a linear damper of damping *c*. The clearance between the two plates of the capacitor in the MEMS is *d*. The coordinate utilized for measuring the displacement of the moving plate is *x*, so the equation of motion for the mechanical resonator would be:

$$m\ddot{x} + c\dot{x} + kx = f \qquad \dot{x} = \frac{dx}{dt} \tag{1}$$

The electric force *f* is

$$f = \frac{\varepsilon_0 A v^2}{2(d-x)^2} = \frac{\varepsilon_0 A}{2(d-x)^2}\left[\frac{1}{2}v_i^2 - \frac{1}{2}v_i^2 \cos(2\omega t)\right] \tag{2}$$

where, ε_0 is permittivity in vacuum. A more detailed calculation of *m*, *c*, and *k* are presented by Raskin et. al. (2000). Introducing a set of variables, we may transform the equation of motion to a nondimentionalized form

$$y'' + hy' + y = \frac{1}{(1-y)^2}\left[\beta - \beta\cos(2r\tau)\right] \tag{3}$$

where,

$$\tau = \omega_n t \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad y = \frac{x}{d} \qquad r = \frac{\omega}{\omega_n}$$
$$h = \frac{c}{\sqrt{km}} \qquad y' = \frac{dy}{d\tau} \qquad \beta = \frac{\varepsilon_0 A}{4kd^3}v_i^2 \tag{4}$$

In this investigation, the model described by Equation (3) has been used and studied. The analysis method is to utilize the nondimensionalized mathematical model of the MEMS and provide key dynamic properties of the system. In case of no polarization voltage, no frequency response can be detected for linearized model, and therefore stability analysis of the system must be done to provide a stability diagram. Provided a stability diagram parameters of the system are possible to be designed to keep the system stable (Nakhaie Jazar 2004, Yu et. al. 2002).

Series expansion indicates that

$$\frac{1}{(1-y)^2} = 1 + 2y + 3y^2 + 4y^3 + 5y^4 + 6y^5 + O(y^6)\,. \tag{4}$$

Assuming $y \ll 1$ and keeping the linear terms of the expansion, generates the following equation of motion.

$$y'' + hy' + \left(1 - 2\beta + 2\beta\cos(2r\tau)\right)y = 2\beta\sin^2(r\tau) \tag{5}$$

This is a forced Mathieu-type differential equation, which has not been analyzed in literatures. Since the parametric excitation, and externally excitation terms in Equation (5) are coupled via excitation frequency *r*, it is not possible to model the system by a Mathieu equation only. Although there is a vast amount research on the Mathieu equation, the stability behavior of the forced Mathieu equation (5) is not referred to.

There are three parameters in the mathematical model of the system, an electric actuation parameter β, the excitation frequency parameter *r*, and the damping parameter *h*. These parameters make a three-dimensional parametric space. Stability character of the system in the parametric space is rendered by determining the stable and unstable domains.

3. MATHEMATICAL ANALYSIS

Following the averaging method we assume a solution for the system, in the following form

$$y = A(\tau)\sin\left(r\tau + \psi(\tau)\right) \tag{6}$$

This solution means

$$\begin{aligned} y' &= A'(\tau)\sin\left(r\tau + \psi(\tau)\right) \\ &+ A(\tau)\left(r + \psi'(\tau)\right)\cos\left(r\tau + \psi(\tau)\right). \end{aligned} \tag{7}$$

However, we may also assume that

$$A'(\tau)\sin\left(r\tau+\psi(\tau)\right)$$
$$+A(\tau)\psi'(\tau)\cos\left(r\tau+\psi(\tau)\right)=0 \quad (8)$$

to have

$$y'=A(\tau)r\cos\left(r\tau+\psi(\tau)\right) \quad (9)$$

Now, substituting (6) and (9) in (5) provides that

$$\begin{aligned}&A'(\tau)r\cos\left(r\tau+\psi(\tau)\right)\\&-A(\tau)r\left(r+\psi'(\tau)\right)\sin\left(r\tau+\psi(\tau)\right)\\&+hA(\tau)r\cos\left(r\tau+\psi(\tau)\right)\\&+\left(1-2\beta+2\beta\cos\left(2r\tau\right)\right)A(\tau)\sin\left(r\tau+\psi(\tau)\right)\\&-2\beta\sin^2\left(r\tau\right)=0\end{aligned} \quad (10)$$

Equations (8) and (10) make a set of equations to be solved for *A(t)* and *ψ(t)*. These equations can be decouple to get the following equations,

$$\begin{aligned}&A(\tau)\psi'(\tau)=\\&\frac{1}{r}\left[2A(\tau)\beta\cos\left(2\varphi(\tau)-2\psi(\tau)\right)\right]\sin^2\left(\varphi(\tau)\right)\\&+\frac{1}{r}\left[A(\tau)\left(1-2\beta-r^2\right)\right]\sin^2\left(\varphi(\tau)\right)\\&+\frac{1}{r}\left[hA(\tau)r\cos\left(\varphi(\tau)\right)\right]\sin\left(\varphi(\tau)\right)\\&+\frac{1}{r}\left[\beta\cos\left(2\varphi(\tau)-2\psi(\tau)\right)\beta\right]\sin\left(\varphi(\tau)\right)\end{aligned} \quad (11)$$

$$\begin{aligned}&A'(\tau)=\frac{1}{r}\cos\left(\varphi(\tau)\right)\times\\&\left[\sin\left(\varphi(\tau)\right)\left(\beta\left(1-2A\right)\cos\left(2\varphi(\tau)-2\psi(\tau)\right)\right)\right]\\&+\frac{1}{r}\cos\left(\varphi(\tau)\right)\left[\sin\left(\varphi(\tau)\right)\left(A(\tau)\left(\beta+r^2\right)\right)-\beta\right]\\&\qquad+hA(\tau)\cos^2\left(\varphi(\tau)\right)\end{aligned} \quad (12)$$

where,

$$\varphi(\tau)=r\tau+\psi(\tau)\,. \quad (13)$$

Assuming $A'(t)$ and $\psi'(t)$ are slow variables and their average remains constant during one cycle provides that we may substitute the right hand sides of Equations (11) and (12) with their integral over one period.

$$\psi'(\tau)=\int_0^{2\pi}\frac{1}{r}\psi'(\tau)d\varphi=\pi\left(1-\beta-r^2-2\beta\cos\left(\psi(\tau)\right)\right) \quad (14)$$

$$A'(\tau)=\int_0^{2\pi}\frac{1}{r}A'(\tau)d\varphi=-\pi A(\tau)\left(hr-\beta\sin\left(2\psi(\tau)\right)\right) \quad (15)$$

3.1. Periodic Response

If there is any possibility to have a periodic steady state response, then $A(\tau)$ and $\psi(\tau)$ must not vary in time. Therefore, the left hand sides of Equations (14) and (15) are zero at steady state conditions.

$$1-\beta-r^2-2\beta\cos\left(\psi\right)=0 \quad (16)$$

$$hr-\beta\sin\left(2\psi\right)=0 \quad (17)$$

Eliminating $\psi(\tau)$ provides a relationship between the parameters of the system to have a periodic steady state response with frequency *r*.

$$r^4-\left(2-h^2-4\beta\right)r^2+3\beta^2-4\beta+1=0 \quad (18)$$

This relationship defines a surface in the parametric space called periodicity surface. The MEMS will vibrate with constant amplitude when the parameters of the system are selected on this surface. Figure 2 depicts a three dimensional view of the periodicity surface, while Figure 3 shows periodicity curves in the parameter plane *r*-β. Both of these figures are valid only for small values of β.

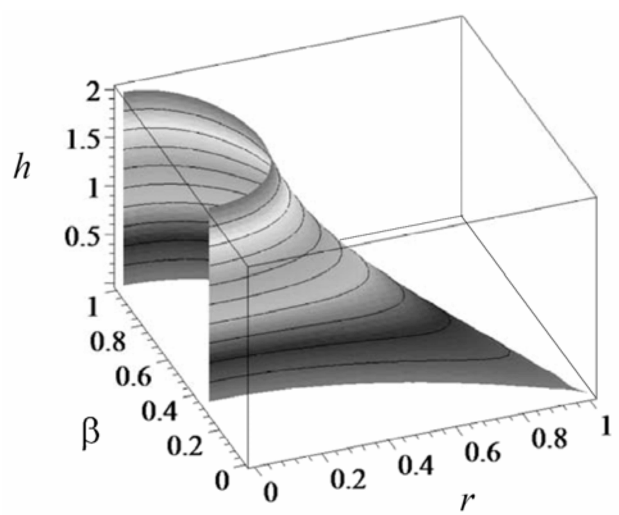

Figure 2. Three dimensional illustration of the periodicity surface for the MEMS.

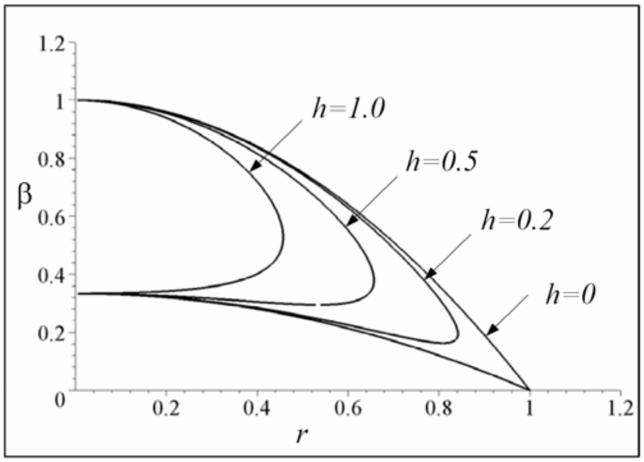

Figure 3. Transient curves and principal instability tongue for MEMS.

Applying the same approach will show that the conditions to have *n*-subharmonics, $n \in N$, and *n*-superharmonics, $1/n \in N$, responses,

$$y = A(\tau)\sin\left(nr\tau + \psi(\tau)\right),$$

is

$$n^4 r^4 - n^2\left(2 - h^2 - 4\beta\right)r^2 + 3\beta^2 - 4\beta + 1 = 0 \qquad (19)$$

The instability tongues for *h=0*, and corresponding to the first three sub and two super harmonics as well as the principal harmonic are shown in Figure 4.

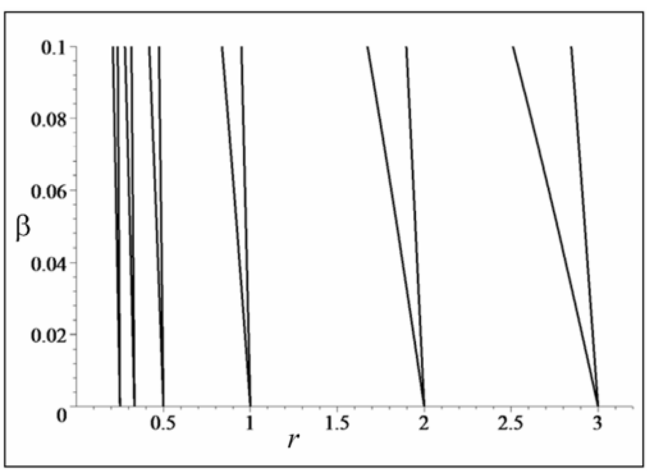

Figure 4. Some of the instability tongues in the parametric plane for MEMS.

In order to determine the stability of the periodic response of the MEMS, we may assign its periodic response with $y_0 = A\sin\left(r\tau + \psi\right)$ and disturb the system as

$$y = y_0 + \delta(\tau) \qquad (19)$$

Substitution of (19) into (5) provides the following equation for evaluation of the disturbance function δ,

$$\delta'' + h\delta' + \left(1 - 2\beta + 2\beta\cos\left(2r\tau\right)\right)\delta = 0 \qquad (20)$$

which is a Mathieu equation. Stability of the periodic response of the MEMS is determined by stability of the Mathieu equation. Applying the averaging method to this Mathieu equation provides exactly the same equation (18) for transient response. Therefore, Figure 3 is a stability diagram and the periodic curves are transient curves, separating stable and unstable regions. Similarly, the periodicity surface of Figure 2 is a transient surface, which separates the unstable interior space, from the stable outer space.

The principal instability region connected to *r=1* is more important, and is what the perturbation method could predict better approximately. So, we concentrate on the principal instability tongue. The transition curve in the stability plane *r*-β may be rewritten in the following form

$$r^2 = \frac{1}{2}\left(2 - h^2 - 4\beta \pm \sqrt{h^4 - 4h^2\left(1 - 2\beta\right) + 4\beta^2}\right). \qquad (21)$$

Since $r^2 \in \mathbf{R}$, then *h* must satisfy the condition

$$2 - 4\beta - 2\sqrt{3\beta^2 + 1 - 4\beta} < h^2 < 2 - 4\beta + 2\sqrt{3\beta^2 + 1 - 4\beta} \qquad (22)$$

to have a transition curve. The transition curves and boundaries of stability for the first instability tongue are plotted in Figure 3 for different value of *h*. As expected, the instability domain shrinks by increasing damping. At each excitation frequency, the domain of stability for β is

$$\begin{aligned}\frac{2}{3}\left(1 - r^2\right) - \frac{1}{3}\sqrt{\left(1 - r^2\right)^2 - 3h^2r^2} \le \beta \\ \beta \le \frac{2}{3}\left(1 - r^2\right) + \frac{1}{3}\sqrt{\left(1 - r^2\right)^2 - 3h^2r^2}\end{aligned}. \qquad (23)$$

3.2. Time Response

Consdering Equations (14) and (15), it is seen that Equation (14) is independent of $A(\tau)$, and can be integrated to obtain the following solution

$$\psi(\tau) = \frac{\pi}{2} + tan^{-1}\left(Z_2 \tan\left(\pi Z_1\left(\tau + C_1\right)\right)\right) \tag{24}$$

where,

$$Z_1 = \sqrt{\left(1 - r^2 - 2\beta\right) - \beta^2} \qquad Z_2 = \frac{Z_1}{1 - r^2 - \beta} \tag{25}$$

and C_1 is the constant of integral.

$$C_1 = \frac{1}{\pi Z_1} tan^{-1}\left(\frac{1}{Z_2} \tan\left(\psi_0\right)\right) \tag{26}$$

Substituting (24) into (15) leads to the following equation

$$\dot{A} + A\left(\frac{2\pi\beta\left(1 - \sqrt{1 + tan^2\left(\psi\right)}\right)tan\left(\psi\right)}{1 + tan^2\left(\psi\right) - \sqrt{1 + tan^2\left(\psi\right)}}\right) + h\pi r A = 0 \tag{27}$$

which can be solve to find $A(t)$,

$$A(\tau) = C_2\left[1 + tan^2\left(\pi Z_1\left(\tau + C_1\right)\right)\right]^{\left(\frac{-\beta Z_2}{Z_1\left(Z_2^2 - 1\right)}\right)} \times\left[1 + Z_2^2 tan^2\left(\pi Z_1\left(\tau + C_1\right)\right)\right]^{\left(\frac{\beta Z_2}{Z_1\left(Z_2^2 - 1\right)}\right)} e^{\left(-hr\pi\left(\tau + C_1\right)\right)} \tag{28}$$

where C_1 is a new constant of integral.

$$C_2 = A_0\left(1 + tan^2\left(C_1 \pi Z_1\right)\right)^{\left(\frac{\beta Z_2}{Z_1\left(Z_2^2 - 1\right)}\right)} \times\left(1 + Z_2^2 tan^2\left(C_1 \pi Z_1\right)\right)^{\left(\frac{-\beta Z_2}{Z_1\left(Z_2^2 - 1\right)}\right)} e^{\left(hr\pi C_1\right)} \tag{29}$$

Therefore, the time response of the system in transient would be

$$y = A(\tau)\sin\left(r\tau + \psi(\tau)\right), \tag{30}$$

where $\psi(\tau)$ and $A(\tau)$ come from Equations (24) and (29) respectively.

Manipulating, the expression of the amplitude in (28) it can be transformed to the following equation

$$A(\tau) = C_2 e^{\left(-hr\pi\left(\tau + C_1\right)\right)} \times \left[\left(\frac{1}{2} - \frac{Z_2^2}{2}\right)\cos\left(2\pi Z_1\left(\tau + C_1\right)\right) + \frac{1}{2} + \frac{Z_2^2}{2}\right]^{\left(\frac{\beta Z_2}{Z_1\left(Z_2^2 - 1\right)}\right)}. \tag{31}$$

Note that since we deleted the fast terms during the averaging process, the analytic solution for the time response is only valid for short durations of time. Figure 5 illustrates a comparison between analytic time response and numerical solution for a set of parameters.

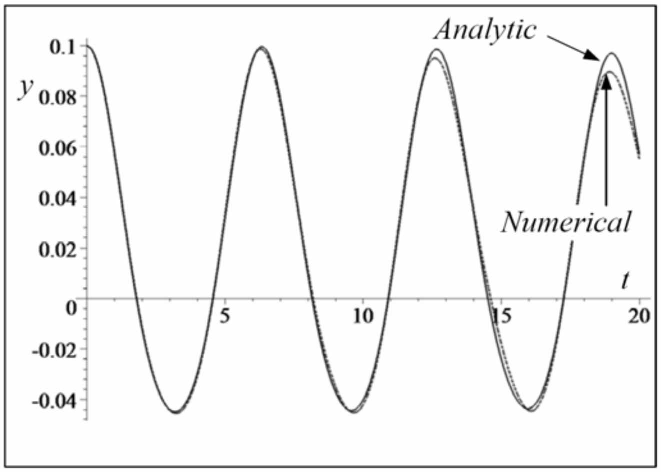

Figure 5. Comparison between analytic and numerical solutions for $h=0.01$, $r=1$, $\beta=0.02$.

Since the equation of motion of the system is linear, there is only one equilibrium position at $y=0$. Eliminating the excitation e.g., $\beta=0$, reduces the system to a simple mass-spring-damper vibrating system which is asymptotically stable, and approaches the equilibrium position when $h>0$. By increasing β, the system may have periodic, unstable or quasi-periodic response depend on the value of excitation frequency r.

In order to investigate the effect of the dynamic parameters on the response of the system, we select a point close to the tip of the principal instability tongue. As indicated by the stability curves, point ($r=0.9$, $\beta=0.04$), and point ($r=1.02$, $\beta=0.04$) are in stable zones, while point ($r=0.97$, $\beta=0.04$) is in unstable zone.

Figure 6 depicts the response of the system for three points around principal resonant and starting from zero initial conditions, to indicate correctness of the stability Figure 3.

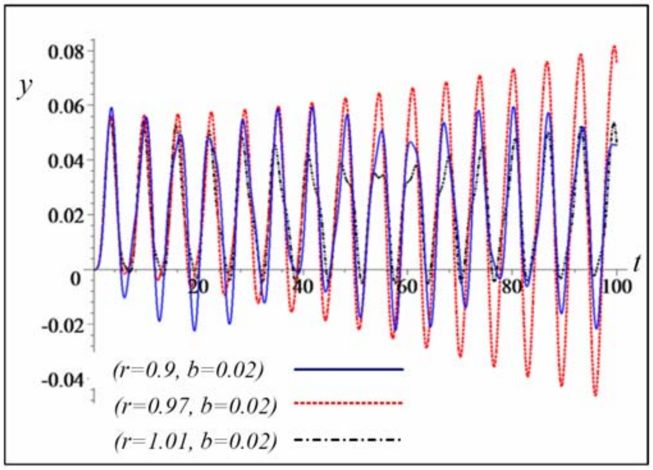

Figure 6. Comparison between three points around the tip of the principal instability tongue for $h=0$.

4. CONCLUSION

The linear model of a MEMS has been developed, and it is shown that its dynamic behavior is governed by an ODE including parametrically and externally excitations. After a simplification by linearizing the restoring and electric forces, a harmonic externally excited Mathieu equation was used to investigate the dynamics of the MEMS. The analytic result based on averaging perturbation method was derived for both steady state and time response of the system. The analytic results show acceptable agreement with numerical integration. The steady state analysis leaded to the conditions for periodic response dividing the stable and unstable regions around the primary resonance. On the other hand, the analytic time response shows the effects of dynamic parameters in behavior of the MEMS, and may be utilized for optimization and design.

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