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COMPARISON OF EXACT AND APPROXIMATE FREQUENCY RESPONSE OF A PIECEWISE LINEAR VIBRATION ISOLATOR

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Abstract

In this paper an investigation is carried out to classify the steady state responses of asymmetric piecewise linear vibration isolators as double hitting, single hitting, and no hitting. In each class, the analysis has been carried out using a set of coupled nonlinear algebraic equations following Natsiavas and Gonzalez [1]. Applying perturbation technique, a closed form analytic expression of the frequency response is also derived for symmetric conditions. The exact frequency response is utilized to validate the analytic results obtained by perturbation techniques. Direct comparison indicates the results obtained by averaging method are mathematically and practically close to the exact solution.

1. INTRODUCTION

Many dynamic systems can be modeled as a one-degree-of-freedom vibrating system possessing piecewise linear or piecewise nonlinear characteristics. A wide range of structural systems of practical interest can be reduced to an equivalent piecewise linear suspension system [2, 3].

The system in this investigation is an asymmetric case of a harmonically base excited single degree-of-freedom vibrating system, which is a model of suspension and vibration isolator with limiting displacement. Figure 1 depicts an illustration of a mechanical model of the system. The primary suspension is permanently attached to the sprung mass *m*, while the positive and negative secondary suspensions interact whenever the relative motion of *m*, with respect to the base is greater than positive clearance, $u > \Delta_P$, or less than the negative clearance, $u < -\Delta_N$.

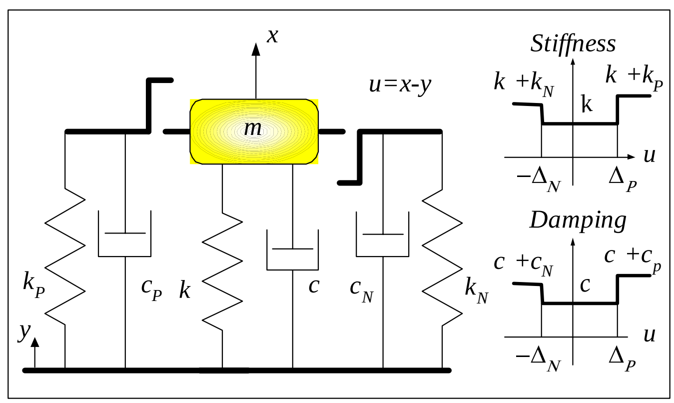

Figure 1. Mechanical model of the system and representation of stiffness and damping rates.

As long as the amplitude of relative motion is less than the absolute value of both clearances, *m* is supported by the primary suspension and the overall stiffness and damping rates are equal to *k* and *c* respectively. As soon as the secondary suspensions interact, the overall stiffness and damping increases to $k+k_P$ and $c+c_P$ in positive direction, and to $k+k_N$ and $c+c_N$ in negative direction. Clearly there could be situations that only one of the secondary suspensions comes into action.

Secondary suspensions are usually added to prevent high relative displacement of the sprung mass. It is known that the acceleration transmissibility of the isolator can be improved by softening the stiffness and damping rates of suspension. However, relative displacement of the system increases by softening the suspension. In such a situation a secondary suspension might be employed to limit the relative displacement and prevent the mass from excessive travel.

2. BACKGROUND

The first study of a piecewise linear system has been done by Den Hartog and Mikina [4] who evaluated the steady state behavior of an undamped system with bilinear stiffness. Later, Den Hartog and Heiles [5] found a closed form solution for frequency response of the same system under harmonic excitation. Gurtin [6] used an equivalent viscous damper and showed the performance of an amplitude-dependant damping in vibration isolation. Masri [7-9] described the difficulties that arise from formulation of the steady state motion; using a general first degree of freedom vibrating system. Later, Natsiavas and Gonzalez [1] solved the problem by seeking for the exact solution to overcome the difficulties involved in approximate methods. Natsiavas [2] could plot the frequency response of the system by applying numerical analysis. Masri and Stott [9] did a sensitivity investigation with an experimental device for few important parameters. Nguyen, et al. [3, 10] utilized numerical integration methods and could describe the time domain behavior of the system.

Natsiavas [11] used perturbation methods and studied the stability of piecewise linear systems, for an assumed steady state periodic response. By applying that method Natsiavas [1] developed the stability analysis for other systems having piecewise linear elements.

Reducing the problem of steady state response of piecewise linear vibrating systems to a set of algebraic equations, and then applying a procedure to solve them numerically is the best applied method up to date. However, the disadvantage of this method is the need of extensive numerical analysis, and lack of a closed form solution.

Jazar and Golnaraghi [12] applied a modified averaging method and derived an approximate analytic solution for frequency response of symmetric bilinear vibration isolators. They expanded their method to investigate kinematics of piecewise linear systems, and could uncover some important properties [13, 14]. Narimani et. al. applying the same method showed that piecewise linear systems have some interesting dynamic behaviors hard to be detected by exact method [15-17].

3. MODELING

The governing differential equation of the system may be presented in the following form

$$\ddot{x} + g_1(x,\dot{x}) = f_1(y,\dot{y}) \tag{1}$$

$$g_1(x,\dot{x}) = \begin{cases} 2\left(\xi\omega_0 + \xi_P\omega_P\right)\dot{x} + \left(\omega_0^2 + \omega_P^2\right)x - \omega_P^2\Delta_P & x - y > \Delta_P \\ 2\xi\omega_0\dot{x} + \omega_0^2 x & -\Delta_N < \left|x - y\right| < \Delta_P \\ 2\left(\xi\omega_0 + \xi_N\omega_N\right)\dot{x} + \left(\omega_0^2 + \omega_N^2\right)x + \omega_N^2\Delta_N & x - y < -\Delta_N \end{cases} \tag{2}$$

$$f_1(y,\dot{y}) = \begin{cases} 2\left(\xi\omega_0 + \xi_P\omega_P\right)\dot{y} + \left(\omega_0^2 + \omega_P^2\right)y & x - y > \Delta_P \\ 2\xi\omega_0\dot{y} + \omega_0^2 y & -\Delta_N < \left|x - y\right| < \Delta_P \\ 2\left(\xi\omega_0 + \xi_N\omega_N\right)\dot{y} + \left(\omega_0^2 + \omega_N^2\right)y & x - y < -\Delta_N \end{cases} \tag{3}$$

where

$$\begin{aligned} 2\xi\omega_0 &= \frac{c}{m} & 2\xi_P\omega_P &= \frac{c_P}{m} & 2\xi_N\omega_N &= \frac{c_N}{m} \\ \omega_0^2 &= \frac{k}{m} & \omega_P^2 &= \frac{k_P}{m} & \omega_N^2 &= \frac{k_N}{m} \end{aligned} \tag{4}$$

Due to piecewise linear characteristic of functions f_1 and g_1, an analytical solution of the system can be obtained for $-\Delta_N < x\text{-}y < \Delta_P$, $x\text{-}y > \Delta_P$, and $x\text{-}y < -\Delta_N$. However, its complete response cannot be reduced to a single explicit function, simply because determination of the instances of switching time at which the relative displacement $z=x-y$ attained the value Δ or $-\Delta$ requires solution of a set of transcendental equations [3]. Having an analytic equation is a big aid in visualizing the characteristics of the system and determining the relationship of dynamical parameters involved.

It is more practical if we transform the equation of motion (1) to the following dimensionless form

$$u'' + g_2(u,u') = f_2(v,v') \tag{5}$$

and then, consider the governing equation in the relative coordinate system $w = u - v$

$$w'' + 2\xi w' + w = -v'' + g_3(w,w') \tag{6}$$

with introducing the following dimensionless variables and parameters.

$$u = \frac{x}{Y} \quad v = \frac{y}{Y} \quad u' = \frac{du}{d\tau} \quad v' = \frac{dv}{d\tau} \quad \tau = \omega_0 t$$
$$\delta_P = \frac{\Delta_P}{Y} \quad \delta_N = \frac{\Delta_N}{Y} \quad r = \frac{\omega}{\omega_0} \quad \rho_P = \frac{\omega_P}{\omega_0} \quad \rho_N = \frac{\omega_N}{\omega_0} \tag{7}$$

$$\dot{x} = \omega_0 Y u' \qquad \dot{y} = \omega_0 Y v' \tag{8}$$

$$g_2 = \begin{cases} 2\left(\xi + \rho_P \xi_P\right)u' + \left(1 + \rho_P^2\right)u - \rho_P^2 \delta_P & u - v > \delta_P \\ 2\xi u' + u & \delta_N < |u - v| < \delta_P \\ 2\left(\xi + \rho_N \xi_N\right)u' + \left(1 + \rho_N^2\right)u + \rho_N^2 \delta_N & u - v < -\delta_N \end{cases} \tag{9}$$

$$f_2 = \begin{cases} 2\left(\xi + \rho_P \xi_P\right)v' + \left(1 + \rho_P^2\right)v & u - v > \delta_P \\ 2\xi v' + v & \delta_N < |u - v| < \delta_P \\ 2\left(\xi + \rho_N \xi_N\right)v' + \left(1 + \rho_N^2\right)v & u - v < -\delta_N \end{cases} \tag{10}$$

$$g_3 = \begin{cases} -2\rho_P \xi_P w' - \rho_P^2 w + \rho_P^2 \delta_P & w > \delta_P \\ 0 & \delta_P < |w| < \delta_N \\ -2\rho_N \xi_N w' - \rho_N^2 w - \rho_N^2 \delta_N & w < -\delta_N \end{cases} \tag{11}$$

4. FREQUENCY RESPONSE

4.1. Exact Method

Applying a harmonic base excitation

$$v = \sin(r\tau) \tag{12}$$

the equation of motion reduces to

$$w'' + 2\xi w' + w = r^2 \sin(r\tau) + g_3(w,w')\,. \tag{13}$$

Due to piecewise linear nature of the function g_3, exact forms of analytical solutions can be obtained for each piece of motion.

Steady state time response of the system might be a double hitting, single hitting, and no hitting as shown in Figure 2 depending on the value of positive and negative gap sizes δ_P, δ_N, and excitation frequency r.

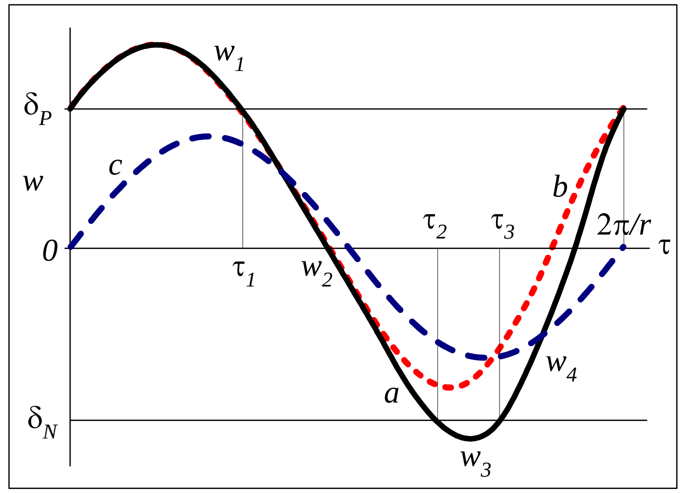

Figure 2. Illustration of steady state time response of piecewise linear suspension system.(*a*, ▬▬▬) double hitting, (*b*, ▪▪▪▪▪) single hitting, and (*c*, ▬ ▬ ▬) no hitting.

When the gap sizes δ_P and δ_N, are set, the existence of a reliable set of solution for the 16 equations depends on the frequency of excitation. The gap sizes δ_P and δ_N may have any value between zero and the tip value of primary frequency response,

$$0 < \delta < \frac{1}{2\xi\sqrt{1-\xi^2}}\,. \tag{14}$$

Intersection of a horizontal line, $w = \delta$ with the frequency response of primary suspension indicate the boundary of frequency domain where the secondary suspensions interact,

$$0 \le r_1 \le r \le r_2 \tag{15}$$

where

$$r_{1,2} = \sqrt{\frac{\delta^2\left(1 - 2\xi^2\right) \mp \delta\sqrt{1 + 4\xi^2\delta^2\left(\xi^2 - 1\right)}}{\delta^2 - 1}} \tag{16}$$

As long as δ_P, and δ_N are less than the limit of Eq. (14) and greater than one, which is the asymptotic value of w when $r \to \infty$, there are four independent frequencies r_{P1}, r_{P2}, r_{N1}, and r_{N2} indicating two hitting frequency ranges, $r_{P2} - r_{P1}$, and $r_{N2} - r_{N1}$. Existence of hitting frequency ranges, $r_{P2}-r_{P1}$, and $r_{N2}-r_{N1}$, provide double hitting and ensure existence of w_1, and w_3. The ranges of hitting frequencies are unbounded when δ_P, and δ_N are less than one. Whenever one of the gap sizes is greater than the limit of Eq. (14), then w_3 disappears and only a single hitting is observed.

When the steady state response of the system is in the class of curve (*a*) shown in Figure 2, then the suspended mass interact with both snubbers. The whole periodic response can be departed into four sections starting at the contact point of the positive snubber. Naming the response of the system in four sections by w_1, w_2, w_3, w_4, and using τ_1, τ_2, τ_3, for the switching times, the solution of equation of motion for each piece, would be

$$w_1 = A_1 e^{z_1\tau} + B_1 e^{z_2\tau} + \frac{Z_3\cos(r\tau+\varphi)+Z_4\sin(r\tau+\varphi)+Z_5}{Z_6} \qquad 0<\tau<\tau_1 \tag{17}$$

$$w_2 = A_2 e^{z_7\tau} + B_2 e^{z_8\tau} + \frac{Z_9\cos(r\tau+\varphi)+Z_{10}\sin(r\tau+\varphi)}{Z_{11}} \qquad \tau_1<\tau<\tau_2 \tag{18}$$

$$w_3 = A_3 e^{z_{12}\tau} + B_3 e^{z_{13}\tau} + \frac{Z_{14}\cos(r\tau+\varphi)+Z_{15}\sin(r\tau+\varphi)-Z_{16}}{Z_{17}} \qquad \tau_2<\tau<\tau_3 \tag{19}$$

$$w_4 = A_4 e^{z_7\tau} + B_4 e^{z_8\tau} + \frac{Z_9\cos(r\tau+\varphi)+Z_{10}\sin(r\tau+\varphi)}{Z_{11}} \qquad \tau_3<\tau<\frac{2\pi}{r} \tag{20}$$

where their time derivatives are

$$\dot{w}_1 = A_1 Z_1 e^{z_1\tau} + B_1 Z_2 e^{z_2\tau} + \frac{-rZ_3\sin(r\tau+\varphi)+rZ_4\cos(r\tau+\varphi)}{Z_6} \qquad 0<\tau<\tau_1 \tag{21}$$

$$\dot{w}_2 = A_2 Z_7 e^{z_7\tau} + B_2 Z_8 e^{z_8\tau} + \frac{-rZ_9\sin(r\tau+\varphi)+rZ_{10}\cos(r\tau+\varphi)}{Z_{11}} \qquad \tau_1<\tau<\tau_2 \tag{22}$$

$$\dot{w}_3 = A_3 Z_{12} e^{z_{12}\tau} + B_3 Z_{13} e^{z_{13}\tau} + \frac{-rZ_{14}\sin(r\tau+\varphi)+rZ_{15}\cos(r\tau+\varphi)}{Z_{17}} \qquad \tau_2<\tau<\tau_3 \tag{23}$$

$$\dot{w}_4 = A_4 Z_7 e^{z_7\tau} + B_4 Z_8 e^{z_8\tau} + \frac{-rZ_9\sin(r\tau+\varphi)+rZ_{10}\cos(r\tau+\varphi)}{Z_{11}} \qquad \tau_3<\tau<\frac{2\pi}{r} \tag{24}$$

and the constant parameters Z_1 to Z_{17} are introduced in the appendix.

In order to find the unknown parameterss $A_1 \ldots A_4$, $B_1 \ldots B_4$, the following boundary conditions must be satisfied.

$$w_1(0)=\delta_P \quad \dot{w}_1(0)=v_1 \quad w_1(\tau_1)=\delta_P \quad \dot{w}_1(\tau_1)=v_2$$

$$w_2(\tau_1)=\delta_P \quad \dot{w}_2(\tau_1)=v_2 \quad w_2(\tau_2)=-\delta_N \quad \dot{w}_1(\tau_2)=v_3$$

$$w_3(\tau_2)=-\delta_N \quad \dot{w}_3(\tau_2)=v_3 \quad w_3(\tau_3)=-\delta_N \quad \dot{w}_3(\tau_3)=v_4$$

$$w_4(\tau_3)=-\delta_N \quad \dot{w}_4(\tau_3)=v_4 \quad w_4\left(\frac{2\pi}{r}\right)=\delta_P \quad \dot{w}_4\left(\frac{2\pi}{r}\right)=v_1 \tag{25}$$

Conditions (25) provide sixteen equations and introduce eight new unknowns. So, a set of sixteen coupled transcendental equations will be set up to provide sixteen unknowns, $A_1 \ldots A_4$, $B_1 \ldots B_4$, v_1, v_2, v_3, v_4, $\tau_1, \tau_2, \tau_3, \varphi$.

These set of algebraic equations can provide the exact amplitude of the steady state response in both, positive and negative directions, upon solution. However, there is no analytic solution and numerical analysis must be employed. Therefore, providing the existence of w_1 and w_3, a set of unknowns will be found by applying a numerical method such as Newton-Raphson. Then the peak values of w_1 and w_3, indicate positive and negative amplitudes. Following this procedure a set of peak values of steady state responses corresponding to a set of excitation frequencies within the range of $r_{P2}-r_{P1}$, and $r_{N2}-r_{N1}$ will be found.

Figures 3(a) and 3(b) depict an example of the steady state response of the system in a double hitting regime illustrating unequal positive and negative frequency responses.

When the steady state response of the system is in the class (b) shown in Figure 4, then the suspended mass interacts with one snubber only. If suspended mass interacts with no snubbers then it acts in the primary frequency response.

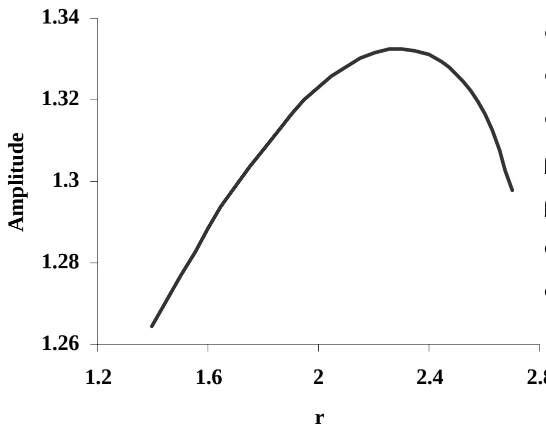

Figure 3(a). Frequency responses of a piecewise linear suspension in positive side

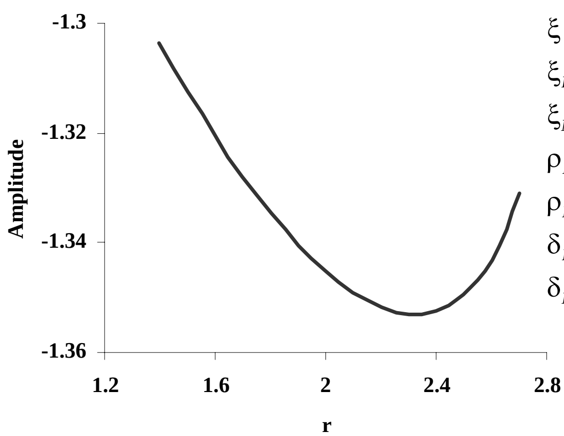

Figure 3(b). Frequency responses of a piecewise linear suspension in positive side

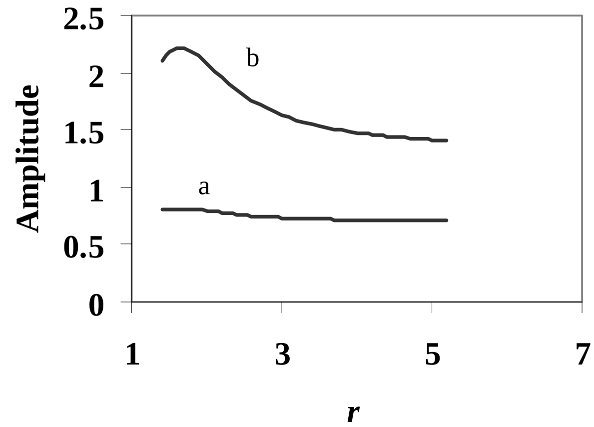

Figure 4. Frequency responses of a suspension with single hit at positive side; (a). Positive amplitude, (b). Negative amplitude

4.2. Averaging Method

Applying a harmonic base excitation

$$v = sin(r\tau) \tag{26}$$

Equation (6) is simplified to

$$w'' + 2\xi_1 w' + w = r^2 sin(r\tau) + g_3(w, w') . \tag{27}$$

For applying the averaging method, the solution of Equation (27) is considered to be

$$w = a(\tau) sin(\varphi(\tau)) \tag{28}$$

where

$$w' = a(\tau) r cos(\varphi(\tau)) \tag{29}$$

$$\varphi(\tau) = r\tau + \beta(\tau) \tag{30}$$

Equations (28) and (29) imply

$$a' sin\varphi + a\beta' cos\varphi = 0 \tag{31}$$

as a constraint equation and provide

$$w'' = a' r cos\varphi - a\beta' r sin\varphi - ar^2 sin\varphi . \tag{32}$$

Using Equations (28), (29), and (32), the Equation of motion (27) leads to

$$\begin{aligned} a' r cos\varphi - a\beta' r sin\varphi &= a(r^2 - 1) sin\varphi \\ &+ r^2 sin(\varphi - \beta) - 2\xi_1 ar cos\varphi + g_3(w, w') \end{aligned} \tag{33}$$

Solving Equations (31) and (33) for α' and β', we have

$$a'r = \left(a\left(r^2-1\right)+r^2\cos\beta\right)\sin\varphi\cos\varphi \\ -\left(2\zeta_1 ar + r^2\sin\beta\right)\cos^2\varphi + g_4(w,w')\cos\varphi \quad (34)$$

$$a\beta' r = \left(-a\left(r^2-1\right)-r^2\cos\beta\right)\sin^2\varphi \\ +\left(2\xi_1 ar + r^2\sin\beta\right)\cos\varphi\sin\varphi - g_4(w,w')\sin\varphi \quad (35)$$

The variables α' and β' are slowly varying with time, and we may assume their average remain constant over a period,

$$2\pi a' r = \int_0^{2\pi} [\left(a\left(r^2-1\right)+r^2\cos\beta\right)\sin\varphi\cos\varphi \\ -\left(2\xi_1 ar + r^2\sin\beta\right)\cos^2\varphi + g_3(w,w')\cos\varphi]\,d\varphi \\ = -2\xi_1 ar\pi - \pi r^2\sin\beta + \int_0^{2\pi}\left(g_3(w,w')\cos\varphi\right)d\varphi \quad (36)$$

$$2\pi\beta' r = \int_0^{2\pi} [\left(-a\left(r^2-1\right)-r^2\cos\beta\right)\sin^2\varphi + \\ \left(2\xi_1 ar + r^2\sin\beta\right)\cos\varphi\sin\varphi - g_3(w,w')\sin\varphi]\,d\varphi \\ = -\pi a\left(r^2-1\right) - \pi r^2\cos\beta - \int_0^{2\pi}\left(g_3(w,w')\sin\varphi\right)d\varphi \quad .(37)$$

where

$$g_3 = \begin{cases} -2\rho\xi_2 ar\cos\varphi & \\ -\rho^2 a\sin\varphi + \rho^2\delta & \varphi_0 < \varphi < \pi - \varphi_0 \\ -2\rho\xi_2 ar\cos\varphi & \\ -\rho^2 a\sin\varphi - \rho^2\delta & \pi + \varphi_0 < \varphi < 2\pi - \varphi_0 \end{cases} \quad (38)$$

and

$$\varphi_0 = \sin^{-1}\left(\frac{\delta}{a}\right). \quad (39)$$

We are interested in steady state response of the system, so the left hand side of Equations (36) and (37) should be considered zero. Therefore,

$$r^2\pi\sin\beta \\ = \left(-2\xi_1 a\pi + 4\rho\xi_2\delta\cos\varphi_0 - 2\rho\xi_2 a\left(\pi - 2\varphi_0\right)\right)r \quad (40)$$

$$r^2\pi\cos\beta \\ = -a\pi r^2 + a\pi\left(1+\rho^2\right) - 2a\varphi_0\rho^2 - 2\rho^2\delta\cos\varphi_0 \quad (41)$$

which can be utilized to get implicit equations for the amplitude a, and phase β as functions of the excitation frequency r.

$$\left(a^2-1\right)\pi^2 r^4 + \left(Z_1^2 - 2aZ_2\pi\right)r^2 + Z_2^2 = 0 \quad (42)$$

$$\beta = \tan^{-1}\left(\frac{Z_1 r}{-a\pi r^2 + Z_2}\right) \quad (43)$$

where

$$Z_1 = \left(-2\xi_1 a\pi + 4\rho\xi_2\delta\cos\varphi_0 - 2\rho\xi_2 a\left(\pi - 2\varphi_0\right)\right) \quad (44)$$

$$Z_2 = a\pi\left(1+\rho^2\right) - 2a\varphi_0\rho^2 - 2\rho^2\delta\cos\varphi_0\,. \quad (45)$$

Equation (42) is the required frequency response function that can be utilized to attain the dimensionless amplitude at frequency r, or vice versa. Figure 5 depicts the frequency response of the system for a piecewise linear vibration isolator compared to a linear isolator, and illustrate the effect of introducing the secondary suspension.

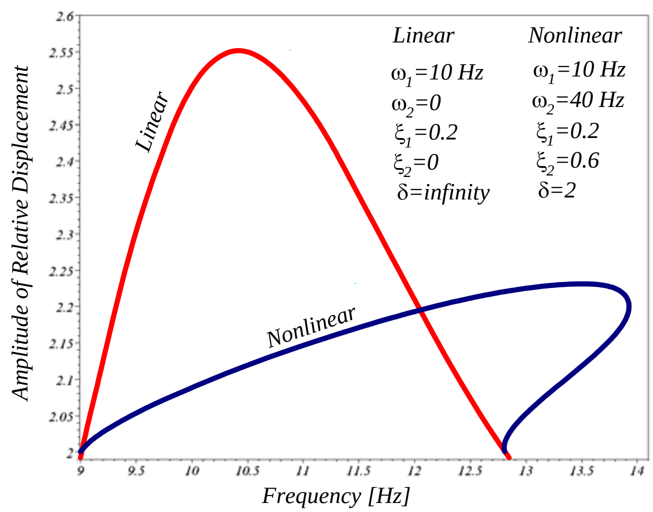

Figure 5. Relative displacement frequency response of piecewise linear system compared with linear system

5. COMPARISONS AND CONCLUSION

In order to compare the result of the averaging method with the exact method, the frequency response of the system is found using both methods. Figure 6 is a plot showing the comparison. The values of parameters are selected to be the same as in Figure 5.

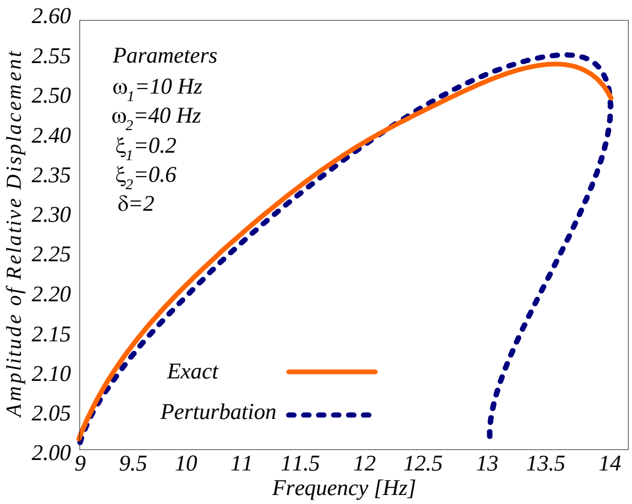

Figure 6. Comparison of the relative displacement frequency response of the piecewise linear system based on exact and perturbation analysis.

It is seen that the result of averaging method matches with the result of exact method. The result of the averaging method deviates from the exact frequency response only a little. Hence, it is reasonable for the designer to substitute one method with the other to simplify the problem. Narimani et al. [16] also have shown that the analytic equation derived by averaging method provides a reasonable agreement with experiments as well as numerical examination.

There are some advantages and disadvantages in using either method. Perturbation method provides an analytic description for the frequency response that can be used for a sensitivity and parameter study. In addition, having a mathematical equation helps the designer to figure out the role of the parameters involved, and analyze thire role in response of the system.

There are five nondimensionalized parameters in the problem. If the primary suspension is already optimized based on some design criterion [18-20], it is supposed that the secondary suspension be designed optimally. Hence, three parameters are remaining. Now the derived equation for the frequency response can be used to optimize the gap size, secondary damping and stiffness, using either gradient methods, or error minimization methods.

The main disadvantage of the perturbation method is that it could not be applied to asymmetric suspensions yet.

The exact method has an obvious advantage that it provides exact frequency response. In addition, the exact method can be applied to asymmetric as well as symmetric piecewise linear suspensions. The exact method can also provide the frequency response of the system in single and double hitting conditions.

There are a few disadvantages involved with the exact method. First, there is no equation to show the effect or contribution of individual parameters in the frequency response of the system. Second, detecting the frequency response means solving a set of nonlinear transcendental equations for a set of parameters and a range of excitation frequencies. Hence, applying the exact method is a tedious and time-consuming task. Some of the equations are multiple valued and the correct answer must be detected based on physical situations. There is still some approximation involved due to numerical solution of the set of equations. Third, the exact method cannot detect odd responses such as frequency islands [17]. Fourth, the exact method fails whenever any nonlinearity is involved. Nonlinearity could be a part of primary or secondary suspensions, or both.

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APPENDIX:

$$Z_1 = -\xi - \rho_P \xi_P - \sqrt{\left(\xi + \xi_P \rho_P\right)^2 - \rho_P^2 - 1}$$

$$Z_2 = -\xi - \rho_P \xi_P + \sqrt{\left(\xi + \xi_P \rho_P\right)^2 - \rho_P^2 - 1}$$

$$Z_3 = -2r^3(1 + \rho_P^2)(\xi + \xi_P \rho_P)$$

$$Z_1 = -\xi - \rho_P \xi_P - \sqrt{\xi^2 + 2\xi\xi_P\rho_P + \rho_P{}^2\xi_P{}^2 - 1 - \rho_P{}^2}$$

$$Z_2 = -\xi - \rho_P \xi_P + \sqrt{\xi^2 + 2\xi\xi_P\rho_P + \rho_P{}^2\xi_P{}^2 - 1 - \rho_P{}^2}$$

$$Z_3 = -2r^3(1 + \rho_P{}^2)(\xi_P\rho_P + \xi)$$

$$Z_4 = -r^2(1 + \rho_P{}^2)(-\rho_P{}^2 - 1 + r^2)$$

$$Z_5 = (\rho_P{}^4 + (2 - 2r^2 + 4\xi_P{}^2 r^2)\rho_P{}^2 + 8\xi\xi_P\rho_P r^2 + 1 + r^4$$
$$+ (4\xi^2 - 2)r^2)\rho_P{}^2\delta_P$$

$$Z_6 = (\rho_P{}^4 + (2 - 2r^2 + 4\xi_P{}^2 r^2)\rho_P{}^2 + 8\xi\xi_P\rho_P r^2 + 1 + r^4$$
$$+ (4\xi^2 - 2)r^2)(1 + \rho_P{}^2)$$

$$Z_7 = -\xi - \sqrt{\xi^2 - 1}$$

$$Z_8 = -\xi + \sqrt{\xi^2 - 1}$$

$$Z_9 = -2\xi r^3$$

$$Z_{10} = -r^2(1-r^2)$$

$$Z_{11} = 1 + r^4 + (4\xi^2 - 2)r^2)$$

$$Z_{12} = -\xi - \rho_N \xi_N - \sqrt{\xi^2 + 2\xi\xi_N\rho_N + \rho_N{}^2\xi_N{}^2 - 1 - \rho_N{}^2}$$

$$Z_{13} = -\xi - \rho_N \xi_N + \sqrt{\xi^2 + 2\xi\xi_N\rho_N + \rho_N{}^2\xi_N{}^2 - 1 - \rho_N{}^2}$$

$$Z_{14} = -2r^3(1+\rho_N{}^2)(\xi_N\rho_N + \xi)$$

$$Z_{15} = -r^2(1+\rho_N{}^2)(-\rho_N{}^2 - 1 + r^2)$$

$$Z_{16} = (\rho_N{}^4 + (2 - 2r^2 + 4\xi_N{}^2 r^2)\rho_N{}^2 + 8\xi\xi_N\rho_N r^2 + 1 + r^4$$
$$+ (4\xi^2 - 2)r^2)\rho_N{}^2\delta_N$$

$$Z_{17} = (\rho_P{}^4 + (2 - 2r^2 + 4\xi_P{}^2 r^2)\rho_P{}^2 + 8\xi\xi_P\rho_P r^2 + 1 + r^4$$
$$+ (4\xi^2 - 2)r^2)(1+\rho_N{}^2)$$