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DISTRIBUTION OF SUB AND SUPER HARMONIC SOLUTION OF MATHIEU EQUATION WITHIN STABLE ZONES

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Abstract

The third stable region of the Mathieu stability chart, surrounded by one π-transition and one 2π-transition curve is investigated. It is known that the solution of Mathieu equation is either periodic or quasi-periodic when its parameters are within stable regions. Periodic responses occur when they are on a "splitting curve". Splitting curves are within stable regions and are corresponding to coexisting of periodic curves where an instability tongue closes. Distributions of sub and super-harmonics, as well as quasi-periodic solutions are analyzed using power spectral density method.

1. INTRODUCTION

Stability diagram for the angular Mathieu equation

$$\ddot{x} + a \cdot x - 2b \cdot x \cdot \cos(2t) = 0 \qquad \dot{x} = \frac{dx}{dt} \qquad (1)$$

is depicted in Figure 1 [1]. The stability diagram is developed by plotting relationship between constant parameters a and b for π and 2π-periodic solutions of Mathieu Equation [2]. The parameters a and b are called control parameters simply because they control dynamic behavior of the equation. More specifically, depending on the value of a and b, response of the equation can be periodic, quasi-periodic, or unstable. The stability chart is made by plotting the curves called transition curves which indicate the boundary of stable and unstable regions [3-10].

In Figure 1, each stable zone is bounded by two transition curves starting off the horizontal axes, from integer roots of equation $a - n^2 = 0$, $n=0, 1, 2, 3,\ldots$. We name the stable zones as *first*, for the zone bounded between transient curves started from $a=0, 1$; *second*, for the zone bounded between transient curves started from $a=1, 4$, and so on. The *third* stable zone, which is bounded between curves starting from $a=4$ and $a=9$, will be examined in this paper as an example.

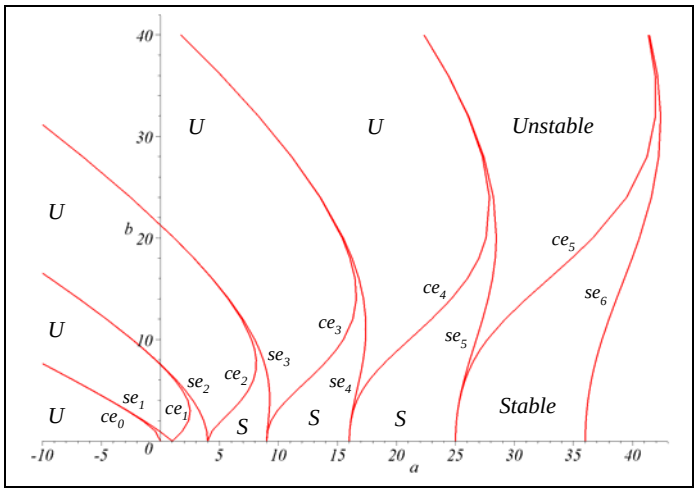

Figure 1. Mathieu stability diagram based on the numerical values, generated by McLachlan (1947).

Power spectral density for time response of the equation is utilized to determine the general behavior of the

system. This investigation also indicates how we may find the splitting (or coexistence) curves within each stable zone. In addition, applying energy-rate method provides a geometrical illustration of the stability within each region.

2. THE THIRD STABLE ZONE

Applying Energy-Rate method [11], an illustration of three dimensional view of 2π-Mathieu stability surface in the third stable zone is shown in Figure 2. Contours of Energy-Rate of the 2π-stability surface are plotted in Figure 3.

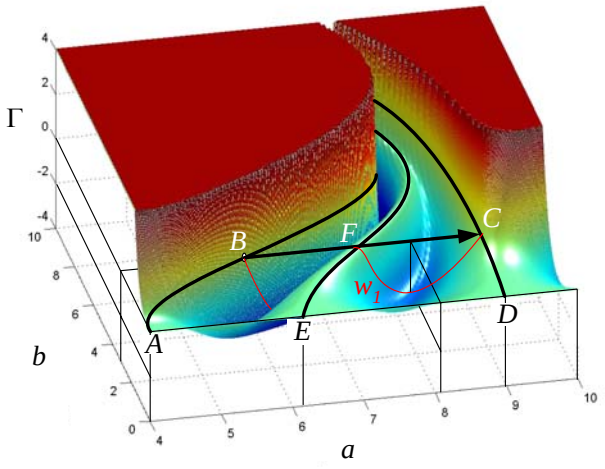

Figure 2. The third stability zone of 2π-stability surface of Mathieu equation

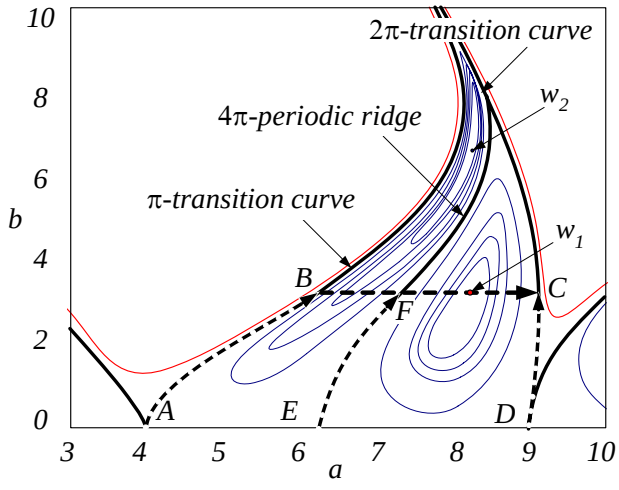

Figure 3. Contours of Energy-Rates in the third stability zone for 2π-stability surface of Mathieu equation

Time integral of an equation of motion over a period for a periodic solution must be zero. Since Mathieu equation has periodic solution on transition curves, its time integral would be zero on these curves, while it is equal to a positive number for a point in unstable region and a negative number for a point within a stable region. Evaluating time integral of Mathieu equation, Γ, over 2π for a domain of parametric plane can determine the stable and unstable regions, as well as splitting lines.

Skew symmetry characteristic of Mathieu equation implies the 2T-periodic transition curves will also appear when Energy-Rate is applied on T-periodic function [11]. Therefore, the ridge in Figure 2 indicates a 4π-periodic splitting curves [9]. As a result, zero level line of the left wall of the stable valley in Figure 2 is a 1 to 1 or π-periodic transition, while the zero level line of the right wall of the third stable valley is a 2 to 1 or 2π-periodic transition curve.

It is seen in Figure 2, that there are two wells w_1, and w_2 in the stable valley at points (a_1=*8.192*, b_1=*3.084*) and (a_2=*8.198*, b_2=*7.04*). The dividing ridge touches the zero level plane and indicates a 4π-periodic curve.

In order to investigate the behavior of Mathieu equation in the third stable region, we will walk along a path from point B on the π-transition wall to point C on 2π-transition wall passing over w_1. We will also analyze the behavior of the equation on the π and 2π-transition curves as well as the 4π-periodic ridge. The two and three dimensional illustration of the paths of investigation are shown in Figures 2 and 3.

Table 1. Characteristics of point 0 to 4

point	*b*	*a*
4	3.084	9.228
3	3.084	8.192
2	3.084	7.205
1	3.084	6.417
0	3.084	6.116

We study the Mathieu equations on 5 different points on the line *b=3.084*, corresponding to the path *B-C*. Point 0 is on the π-transition curve, while points 1 is an arbitrary investigating points between the π-transition curve and the 4π-periodic ridge at local minimum. Point 2 is on the

4π-periodic ridge. Point 3 is corresponding to the well w_1, and point 4 is an investigating point on the 2π-transition curve of Mathieu stability diagram. The information and characteristics of points 0 to 4 are summarized in Table 1.

3. ANALYSIS

Phase plane, time response, and power spectral density of Mathieu equation for parameters at points 0 to 4 are plotted in Figure 4(a) to Figure 4(e), applying initial conditions $x(0)=1, \dot{x}(0)=0$. The powers are calculated based on fast Fourier transform algorithm over 20π time response [12]. Time history responses are plotted for the first 10π time units, while phase plane responses are showing for 4π time units. In addition, the first four Poincare points are also indicated on the phase plane responses for shooting time T_p=π.

Figure 4(a) indicates that point 0 has a π-periodic response based on time response, Phase plane, and repeating Poincare points. In addition, power density designates a strong π-periodic spike as well as a (π/2)-periodic and (π/3)-periodic sub-harmonics. It also represents existence of a very long period super-harmonic. The power of π-periodic response is much higher than sub-harmonics, indicating that π-periodic response is the dominant period, but it is an asymmetric oscillation.

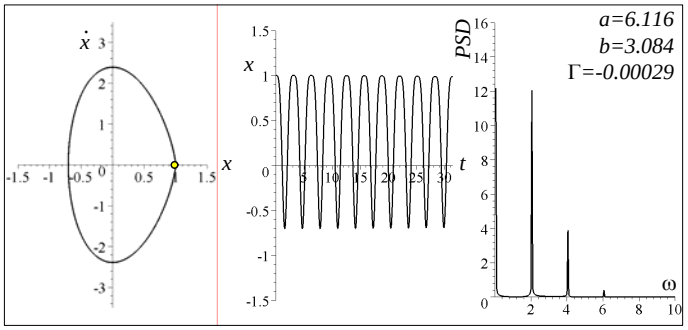

Figure 4(a). Phase plane, time response, and power of Mathieu equation of point 0 on path BC.

In Figure 4(b), *a* is *6.417* corresponding to the local minimum of Energy-Rate at point 1. The response of the system at this point is quasi-periodic response with an overall period very close to 10π as indicated in phase plot of the system. Point 2 at *a=7.205* corresponds to the 4π-periodic ridge.

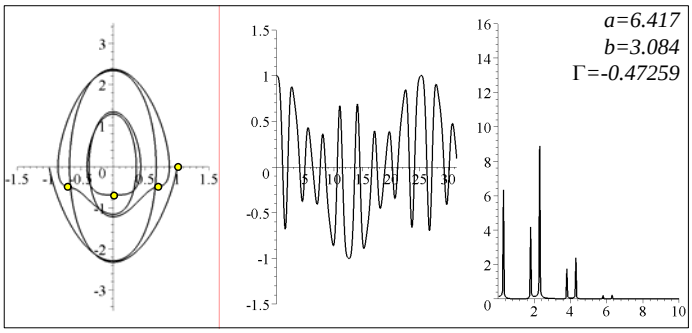

Figure 4(b). Phase plane, time response, and power of Mathieu equation of point 1 on path BC.

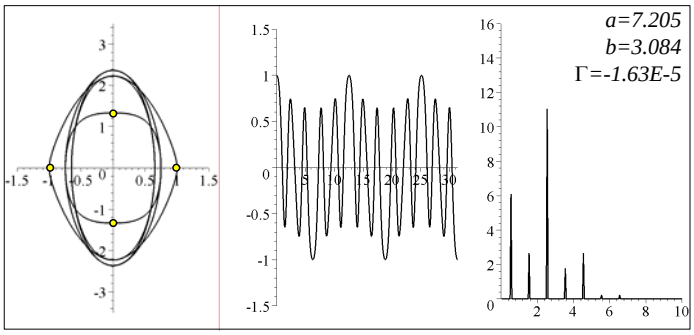

Figure 4(c). Phase plane, time response, and power of Mathieu equation of point 2 on path BC.

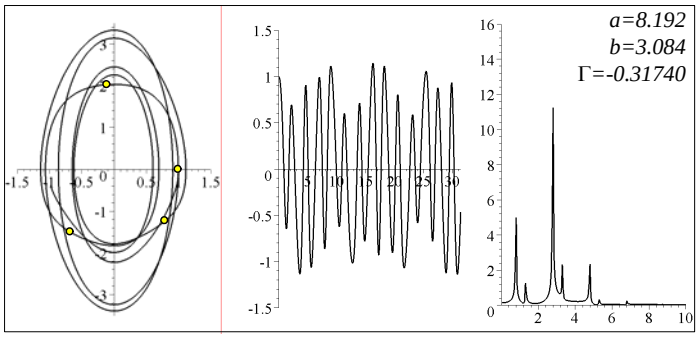

Figure 4(d). Phase plane, time response, and power of Mathieu equation of point 3 on path BC.

The 4π-periodic response is obvious by investigating the phase plot, Poincare points, and time response of the system in Figure 4(c). At this point, the spikes of the power spectral appear exactly at points *0.5, 1.5, 2.5, 3.5, 4.5, 5.5, 6.5*, which are corresponding to (4π), (4π/3), (4π/5), (4π/7), (4π/9), (4π/11), and (4π/13) periods. The power of (4π), and (4π/5) are stronger than the other sub-

harmonics, while (4π/11), and (4π/13) sub-harmonics are the weakest.

Figure 4(d) illustrates the behavior at the well w_1, and Figure 4(e) depicts response of the system at point 4 corresponding to 2π-transition curve.

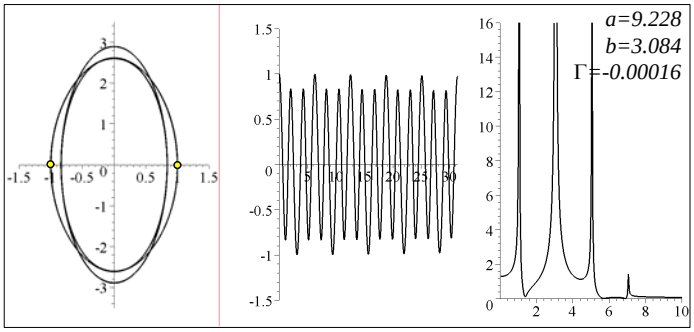

Figure 4(e). Phase plane, time response, and power of Mathieu equation of point 4 on path BC.

Investigating powers in Figures 4(a) to 4(e) presumes that by moving from π-transition curve to 2π-transition curve, the period must gradually change from the period set {(π), (π/2), (π/3), (π/4), …} to the set {(2π), (2π/3), (2π/5), (2π/7), …}. Figure 5(a) depicts the power spectral density of Mathieu equation when the investigating point moves on line BC slowly. The resolution of the graph is 10000 points per unit *a*. It clearly shows the variation of super and sub harmonics of Mathieu equation in third stable region, for *b=3*.084, and *6.116<a<9*.228. It also indicates the power of sub-harmonics are much lower than the power of harmonics around (π), and (π/2). Projection of the power surface makes a two dimensional graph indicating the peak values of power on (*a*, ω) plane, as shown in Figure 5(b).

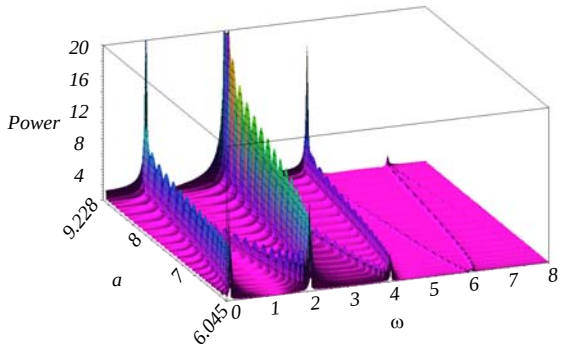

Figure 5(a). 3-dimensional illustration of power spectral density of Mathieu equation in third stable region, for *b=3*.084.

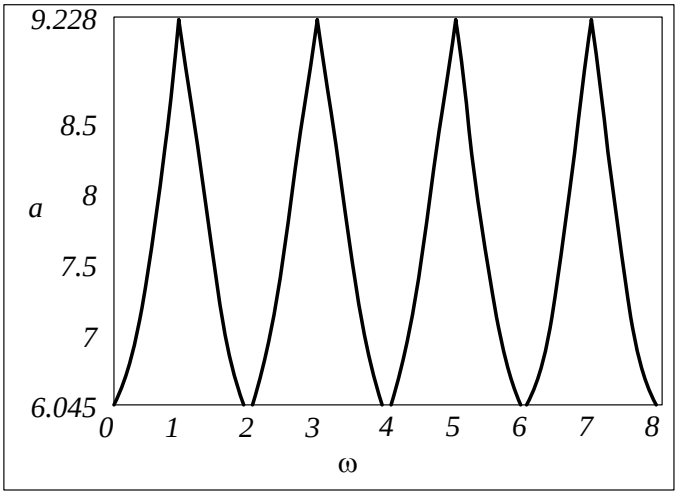

Figure 5(b). Top view of power spectral density surface of Mathieu equation in third stable region, for *b=3*.084.

Response of Mathieu equation for *b=0* must be a simple harmonic oscillation with a unique principal period equal to $T = 2\pi/\sqrt{a}$, while its response is periodic or quasi-periodic for $b \neq 0$. In order to investigate distribution of sub and super harmonics of Mathieu equation on transition curves and on a periodic ridge, we illustrate power spectral density of the equation on the lines AB, DC and EF of Figure 2 or 3. Line AB is on the π-periodic ce_2 (cosine elliptic of second order), line DC is on the 2π-periodic se_3 (sine elliptic of third order), and line EF is on the 4π-periodic ridge.

A 3-dimensional illustration of power spectral density of Mathieu equation on the π-transition curve, between points A and B is depicted in Figure 6. Obviously, the response of Mathieu equation at point A must be a pure π-periodic response with no sub or super harmonics. But when $b \neq 0$, sub and super harmonics appear, although *a* and *b* are set on a transition line. A unique and alone π-periodic response at point A is obvious in Figure 6. By moving from A toward B, a very long super-harmonic, and (π/2), (π/3), (π/4), …, sub-harmonics are appearing and growing. The power of π-periodic response is much higher than the other harmonics while *b* is sufficiently small. The power of π-periodic response has a negative rate, while the long super-harmonic and sub-harmonics have positive rates, indicating that participation of super and sub-harmonics are more at higher values of *b*.

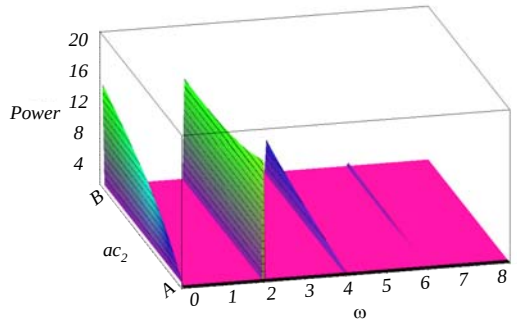

Figure 6. Three-dimensional illustration of power spectral density of Mathieu equation on the π-transition curve ce_2, between points A and B of Figure 3.

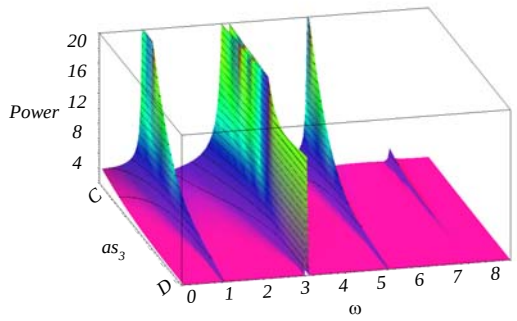

Figure 7. Three -dimensional illustration of power spectral density of Mathieu equation on the π-transition curve se_3, between points C and D of Figure 3.

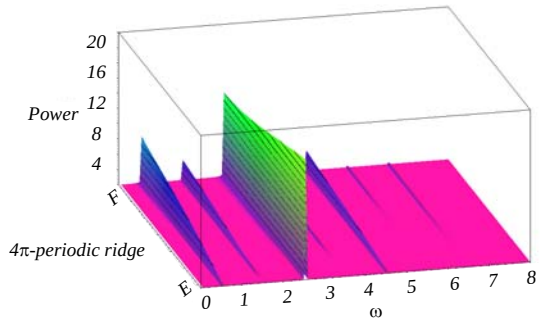

Figure 14. Three -dimensional illustration of power spectral density of Mathieu equation on the 4π-periodic ridge, between points E and F of Figure 3.

A power spectral density on 2π-transition curve, between points D and C is illustrated in Figure 7. Once again, the equation has a pure harmonic response with period $T=2\pi/3$ at point D, but when we move toward point C, sub and super-harmonics are added to the response, as shown in Figure 7. Power spectral in Figure 7 also indicates that sensitivity of Mathieu equation to variation of *a* is getting higher at high values of *b*. In fact, detecting the transition curves of Mathieu equation at high values of both *a* and *b*, are not easy because of increasing sensitivity.

The first point E, of the 4π-periodic ridge EF is at $a=6.25$ providing a simple harmonic motion with period $T=4\pi/5$. By moving up, power spectral graph shown in Figure 8, show appearance of super and sub harmonics at (4π), (4π/3), (4π/5), (4π/7), (4π/9), (4π/11), and (4π/13). By increasing *b*, the power of 4π-harmonic is growing faster than sub-harmonics, while the power of (4π/5)-harmonic has a decreasing trend.

CONCLUSION

The third stable region of the Mathieu stability diagram has been investigated as an example of the stable regions. It is shown that the 2π-periodic Energy-Rate surface shows one 4π-periodic ridge as a coexistence splitting curve. The existence of super and sub-harmonics in quasi-periodic response of Mathieu equation has been analyzed on transition curves and on the 4π-splitting curve of the third stable region, using power spectral density.

The Mathieu equation reduces to a simple harmonic oscillator with period $T=\sqrt{a}=n,\ n\in N$, when $b=0$. For nonzero values of *b*, the response of Mathieu equation is quasi-periodic. When the values of *a* and *b* are such that the corresponding power plot indicates spike at $\omega=m/k,\ \{k,m\}\in N$, then the point *(a, b)* is located on a periodic curve starting from $a=\left(\frac{nk+m}{k}\right)^2,\ \{k,m\}\in N,\ k\geq 2,\ m<k$. The periodic curve is a transition curve, if *k* mod *m*, otherwise it is a splitting curve. The splitting curves have periods $T=2k\pi/m$.

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