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ANALYSIS OF SOLITARY WAVES IN ARTERIES

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ABSTRACT

Solitary waves are coincided with separaterices, which surrounds an equilibrium point with characteristics like a center, a sink, or a source. The existence of closed or spiral orbits in phase plane predicts the existence of such an equilibrium point. If there exists another saddle point near that equilibrium point, separatrix orbit appears.

In order to prove the existence of solution for any kind of boundary value problem, we need to apply a fixed-point theorem. We have used the Schauder's fixed-point theorem to show that there exists at least one nontrivial solution for equation of wave motion in arteries, which has a spiral characteristic.

The equation of wave motion in arteries has a nonlinear character. Thus, the amplitude of the wave depends on the wave velocity. There is no general analytical or straightforward method for prediction of the amplitude of the solitary wave. Therefore, it must be found by numerical or non-straightforward methods. We introduce and analyse three methods: saddle point trajectory, escape moving time, and escape moving energy. We apply these methods and show that the results of them are in agreement, and the amplitude of a solitary wave is predictable.

KEYWORDS

Solitary waves, Fixed point theorems, periodic solutions, separatrix detection

INTRODUCTION

The solution of the type $u=u(z)$ where $z=x+vt$, $v=cte$ indicates stationary travelling waves. After this substitution, the wave equations if are in partial derivatives, are transformed to ordinary differential equations. If the final equations are autonomous, one can use the method of phase trajectories, but the phase space of these equations are degenerated due to the wave velocity, v. Therefore, all singularities, trajectories, separatrices, limit cycles, ..., form a continuum. In addition, one can also see the possible closed separatrix exists only at positive wave velocity. In general, determining in what degree the properties of stationary solutions, including solitons, depend on wave velocity is a complicated problem. The separatrix are those trajectories which are going out of a saddle and returning to it (homoclinic) or entering another saddle (hetroclinic), [1]. Solitary waves involve single-wave pulses with a bell-shaped profile propagating with constant speed.

It was Russell who discovered the solitons in 1834, [2], and finally showed the scientific importance of solitary waves, [3]. Boussinesq in 1872, [4] and Rayleigh in 1876, found independently the hyperbolic secant-squared solution (solitons) for the free surface. Boussinesq found a solution from the approximation of the wave equations, which is now named after him. Again in 1895, Korteweg and deVries found the unidirectional equation of solitaries, which is now named after them. In less than 5 decades latter, soliton had been found in many other branches of science, [5].

The wide application of methods developed in the theory of oscillations and the wave theory were due to the progress in radio physics, plasma physics, and laser optics. In these applications, the most important problems were related to nonlinear waves in depressive media. Therefore almost all the basic oscillation concepts, such as phase plane, self excitation, limit cycle, bifurcation, resonance, ..., have been widely used in the theory of waves. This was naturally accompanied by an intense development of approximate methods.

The method of phase plane was introduced to the wave theory in the early sixties, and very soon was widely used for analysing the behaviour of shock waves, envelope waves and other types of solutions. Separaterices deserve special attention among phase trajectories on the wave phase plane. They illustrate the distinction in the roles of analogous types of solutions for the cases of oscillations and waves. A separatrix is a normalizable solution between the regions of phase space with topologically different types of trajectories, [6].

In a resent paper, Epstein and Johnston, presented a numerical scheme for predicting the amplitude of solitary waves in an elastic artery with any given speed of wave, and vice versa, [7]. In this paper we show that the equation of motion has nontrivial solution, and therefore the past efforts on extracting quantitative and qualitative information is acceptable. With a phase plane analysis, we show that the domain of periodic solution is surrounded with a separaterix. The separaterix is coincident with the desired solitary wave. Then, the amplitude of the solitary wave will be found with alternative methods.

NOMENCLATURE

a, b, c, k, ω	constant parameters
E	escape moving energy
f	function
h	tube wall thickness
I_1, I_2	invariants of the Green deformation tensor
p	total inner pressure
R	initial radius of tube
s	dummy variable
s_z, s_θ	stress components
S, B	Banach spaces
t	time
u	radial displacement
U	operator
v	wave velocity
z	axial coordinate
α, m	material constants
Γ	escape moving time
λ_z	axial stretch ratio
λ_θ	circumferential stretch ratio
Λ_z	stretch in axial direction
Λ_θ	stretch in circumferential direction
ρ	mass density of tube material
ρ_f	mass density of fluid
σ_z, σ_θ	total Cauchy stresses
μ	shear modulus of tube material
Σ	strain energy
$()'$	$\partial()/\partial z$

subscripts

m	maximum
i	inflection
I	initial
z	axial
θ	circumferential
1, 2	invariant indication.

MATHEMATICAL ANALYSIS

Consider the following general second order ordinary differential equation,

$$u'' = f(u,u') \tag{1}$$

where f is a continuous real-valued function with domain R^2. It is smooth enough to ensure existence and uniqueness of solution with any set of initial conditions. Now we establish the following theorem.

THEOREM: If there exist constants a and b, $a \geq b$, such that,

$$f(a,0) \leq 0 \leq f(b,0) \tag{2}$$

then there exists at least one ω such that equation (1) has a nontrivial solution satisfying the following boundary condition,

$$u(0) = u(\omega) \tag{3}$$

PROOF: Let

$$N = max\{|a|,|b|\} \tag{4}$$

and $q > N$ be an arbitrary constant such that

$$3M < k(q-N) \tag{5}$$

where $k > 0$ and

$$M = max\left\{ \left|ku + f(u,u')\right| : (u,u') \in Z \right\} \tag{6}$$

$$Z = max\left\{ (u,u') \in R^2 : |u-u_i| \leq 2q, |u'| \leq 2q\sqrt{k} \right\} \tag{7}$$

where $b \leq u_i \leq a$ and $f(u_i, u'_i) = 0$.

Now, for each c in the interval $[b, a]$, we define the function

$$u(z) = u(z,c) \tag{8}$$

as the solution of the following integral equation

$$u(z) = c\cos(\sqrt{k}z) + q\,sgn(c)\sin(\sqrt{k}z) + F\left(z,u(z),u'(z)\right) \tag{9}$$

where,

$$F\left(z,u(z),u'(z)\right) = \frac{1}{\sqrt{k}}\int_0^z \sin\left(\sqrt{k}(z-s)\right)\left(ku(s) + f\left(u(s),u'(s)\right)\right)ds \tag{10}$$

It can be shown that nontrivial defined function $u(z)$ in (9) satisfies equation (1).

Let S be a Banach space

$$S=\left\{u(z)\in C^1\left(0,\frac{3\pi}{2\sqrt{k}}\right)\right\} \tag{11}$$

One may define a subspace B of the Banach space S which is a closed convex set,

$$B=\left\{u(z)\in C^1\left(0,\frac{3\pi}{2\sqrt{k}}\right),|u(z)-u_i|\le 2q,|u'(z)|\le 2q\sqrt{k}\right\} \tag{12}$$

moreover, the continuous operator U on B, could be defined as,

$$U(u(z))=c\cos(\sqrt{k}z)+q\,sgn(c)\sin(\sqrt{k}z)+F(z,u(z),u'(z)) \tag{13}$$

Then we have,

$$|U(u(z))|\le|c|+|q|+\frac{3M}{k} \tag{14}$$

$$|U'(u(z))|\le\sqrt{k}\left(|c|+|q|+\frac{3M}{k}\right) \tag{15}$$

It is seen that U maps S continuously into itself due to the assumption (5). A mapping from Banach space into another Banach space or into the real numbers is called continuous if sufficiently small change in the argument produces arbitrarily small changes in the function values. It is known, by Schauder's fixed point theorem, that if a completely continuous operator, U, maps the closed convex subspace B, of a Banach space S, into itself, then there exists a fixed point in this mapping, which in turn is a point $u\in B$ such that $U(u)=u$, [8]. Thus, (1) has a nontrivial solution.

In addition since for any $u\in B$, $|F(z,u,u')|\le\dfrac{3M}{k}$, we have

$$\left[c-q\,sgn(c)-F\left(\frac{\pi}{2}\sqrt{k},u(\frac{\pi}{2}\sqrt{k}),u'(\frac{\pi}{2}\sqrt{k})\right)\right]\cdot\left[c+q\,sgn(c)-F\left(\frac{3\pi}{2}\sqrt{k},u(\frac{3\pi}{2}\sqrt{k}),u'(\frac{3\pi}{2}\sqrt{k})\right)\right]<0 \tag{16}$$

So, there exists at least an $\omega\in\left(\frac{\pi}{2}\sqrt{k},\frac{3\pi}{2}\sqrt{k}\right)$ such that

$$c=c\cos(\sqrt{k}\omega)+q\sin(\sqrt{k}\omega)+F(\omega,u(\omega),u'(\omega)) \tag{17}$$

which means $u(0)=u(\omega)$.

This theorem shows that the point $u=u_i$ is an equilibrium point of the system and it can be a center, sink, or source. If in addition, $f(0,0)=0$, and the point $u=0$ is a saddle point, then the system could have separaterix, which originates at $u=0$, surrounds $u=u_i$ and terminates at $u=0$.

MOTION ANALYSIS

It was shown that the equation of motion for wave propagation through fluid-filled elastic is,

$$u''=\frac{m}{s_z-mv^2}\left(\frac{s_\theta}{1+u}-\frac{1}{2}(p_0+p)(1+u)\right) \tag{18}$$

where,

$$p=\frac{v^2}{2}\left(1-\frac{1}{(1+u)^4}\right),\quad p_0=2s_\theta|_{u=0} \tag{19}$$

Equation (18) is an approximation in the sense that the amplitude and slope is assumed small everywhere, so $u<<1$, and terms proportional to $(\partial u/\partial z)^2$ could be neglected, [7]. The explicit form of the functions s_0, s_z depend on the particular constitutive equation for the tube material. They are given by Demiray, [9] as:

$$s_\theta=\left[\lambda_\theta^2(1+u)^2-\frac{1}{\lambda_\theta^2\lambda_z^2(1+u)^2}\right]e^{\alpha(I_1-3)} \tag{20}$$

$$s_z=\left[\lambda_z^2-\frac{1}{\lambda_\theta^2\lambda_z^2(1+u)^2}\right]e^{\alpha(I_1-3)} \tag{21}$$

$$I_1=\lambda_\theta^2(1+u)^2+\lambda_z^2+\frac{1}{\lambda_\theta^2\lambda_z^2(1+u)^2} \tag{22}$$

Now it is seen that the equation of motion (16) is in the form of equation (1), and figure 1 shows that the condition (2) is fulfilled for the following values, which are in the order of magnitude of actual biological measurements, [10].

$$\alpha=1.948\quad m=0.4\quad \lambda_\theta=1.2\quad \lambda_z=1.5\quad v=8 \tag{23}$$

If we choose $k=1$, then it is seen that,

$$a>0,\ b>0,\ N=max\{|a|,|b|\}=a<u_m \tag{24}$$

where $u_m\approx 0.063869$. The graph of $u+f$ is shown in figure 2. It is seen that,

$$M=max\{|u+f(u,u')|\ :(u,u')\in Z\}\approx 0.070268 \tag{25}$$

Thus, if q is any number satisfying the following inequality

$$q>3M+a$$

then there exists at least one initial condition in the following rectangle, which satisfies the condition (3).

$$| u - u_i | \leq 2q \quad , \quad | u' | \leq 2q$$

Every function which satisfies the condition (3) could have periodic solution; at least in a bounded domain of initial conditions.

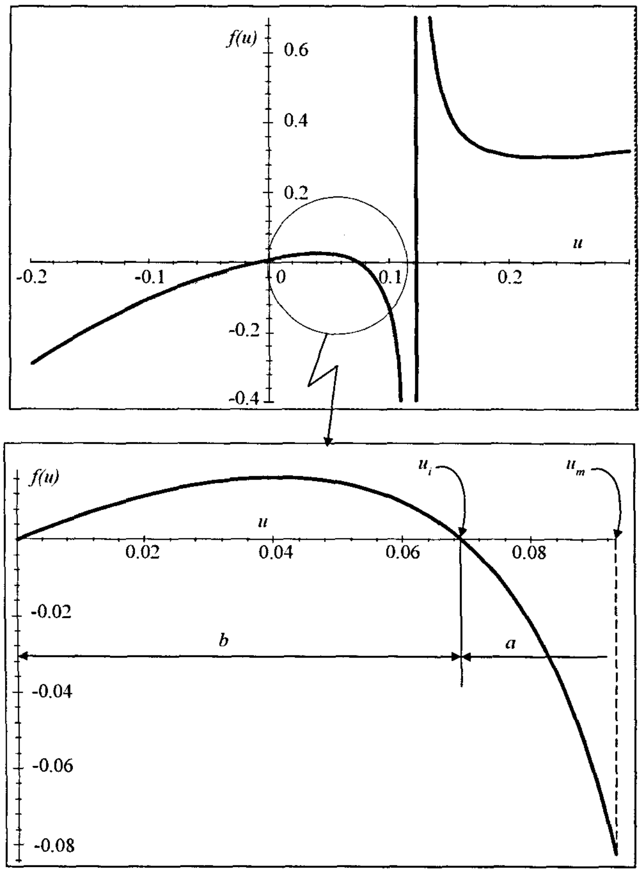

Fig. 1- right hand side of equation (16)

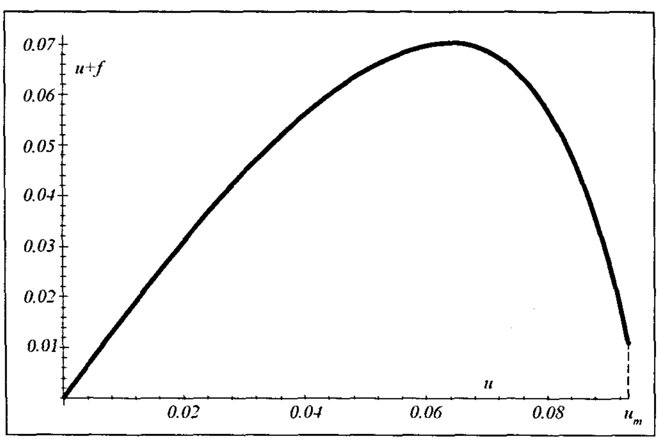

Fig. 2- illustration of $u{+}f$

NUMERICAL ANALYSIS

PHASE PLANE METHOD

In this section we are going to analyze the system in phase plane and also look for some numerical method to evaluate the amplitude u_m of the solitary wave.

The method of phase plane was introduced to the wave theory in the early sixties, and very soon was widely used for analysing the behaviour of shock waves solitons, envelope waves and other types of solutions. Separaterices deserve special attention among phase trajectories on the wave phase plane. They illustrate the destination in the role of analogous types of solutions for the cases of oscillations and waves. A separaterix is a normalizable solution between the regions of phase space with topologically different types of trajectories, [11].

The phase portrait and tangent field of equation (16) is shown in figure 3. The two orbits that are plotted in solid line, pass through points (0.091,0) and (0.095,0). Their time histories are also illustrated in figure 4.

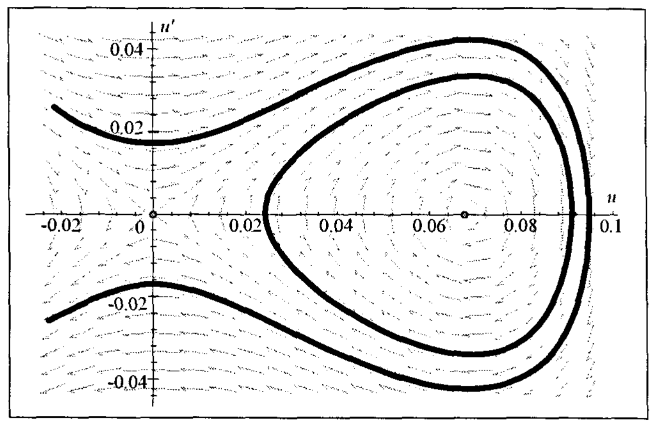

Fig. 3- phase portrait and tangent field of equation (16)

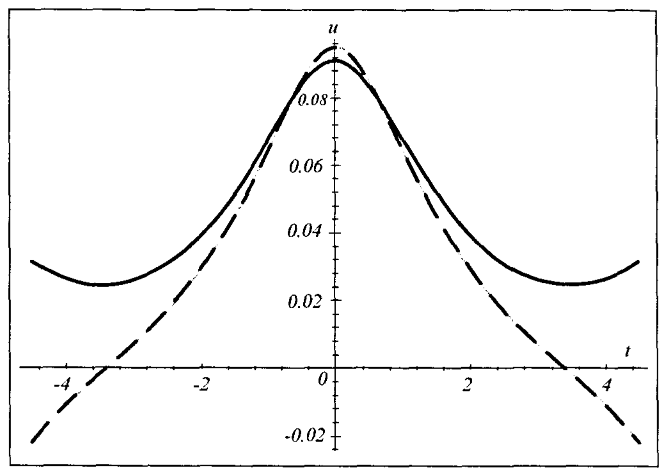

Fig. 4- time history of the orbits shown in Fig. 3

It is seen from figure 3 that the origin is a saddle point and $(u_i, 0)$ is a center. It would be predictable by considering figure 1, as well. The saddle point trajectory separates two different motions, and indicates the solitary wave. Here the saddle point orbit is a homoclinic one since it leaves the equilibrium and returns asymptotically to it as time increases.

We would like to find the amplitude, u_m, of the solitary wave, which is the point of intersection of the saddle point orbit with axis u.

It is known that if the equation of motion is in the form of $u'' = f(u)$, then the integral of $f(u)$ between 0 and u_m must vanish, [7]. Thus u_m, which is shown in figure 1, could be found by satisfying the following equation:

$$\int_0^{u_m} f(u)\,du = 0 \tag{26}$$

The solitary wave must sit between orbits of figure 3 and or graphs of figure 4. The time history of the solitary wave is illustrated in figure 5.

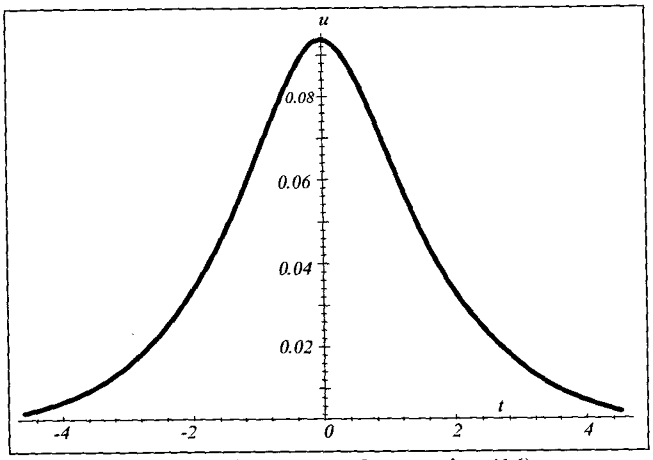

Fig. 5- solitary wave for equation (16)

In order to find the amplitude of the solitary wave, u_m, we are looking for the saddle point trajectory. To do this, we linearize the state equations around the saddle point, $(0, 0)$.

$$\begin{aligned} \dot{u} &= y \\ \dot{y} &= \left.\frac{\partial f}{\partial u}\right|_{(0,0)} u + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y = 0.68147\,u \end{aligned} \tag{27}$$

Thus, the eigenvalues and eigenvectors of the system are

$$\pm 0.8255121 \qquad \begin{Bmatrix} 0.7711793774 & -.7854004566 \\ 0.6366179135 & 0.6483575866 \end{Bmatrix} \tag{28}$$

We are only concerned with the positive half space $u>0$, due to physical appearance of the waves in arteries. Now it is seen that one of the eigenvectors shows the direction of departure from saddle point, and the other one shows the direction of arrival to the saddle point. We call the first one, positive eigenvector and the later, negative eigenvector.

Now we can disturb the states of the system from unstable saddle point equilibrium on the positive eigenvector direction and set the system to be released from following initial conditions:

$$\begin{aligned} u(0) &= 0.00001 \times 0.7711793774 \\ y(0) &= 0.00001 \times 0.6366179135 \end{aligned} \tag{29}$$

Then, the states of the system will change due to the nonlinear equation of motion, and will trace the saddle point trajectory. The intersection of the trajectory with u axes will determine the amplitude of the solitary wave u_m. Figure 6 shows the saddle point orbit, and its time history is illustrated in figure 7. It is seen that the amplitude of the wave is approximately equal to $u_m = 0.063869$ again.

Disturbance of a system from a saddle point equilibrium in direction of positive eigenvector in order to find the approximate amplitude of the solitary wave, is not only applicable to equation (18), but also to any equation of the form $u'' = f(u, u')$.

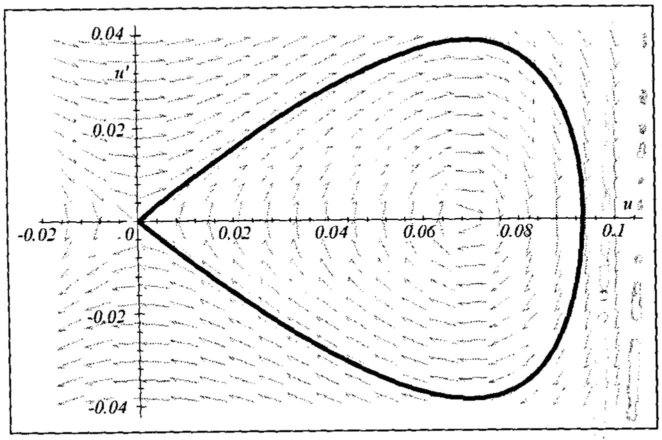

Fig. 6- phase plane illustration of saddle point orbit

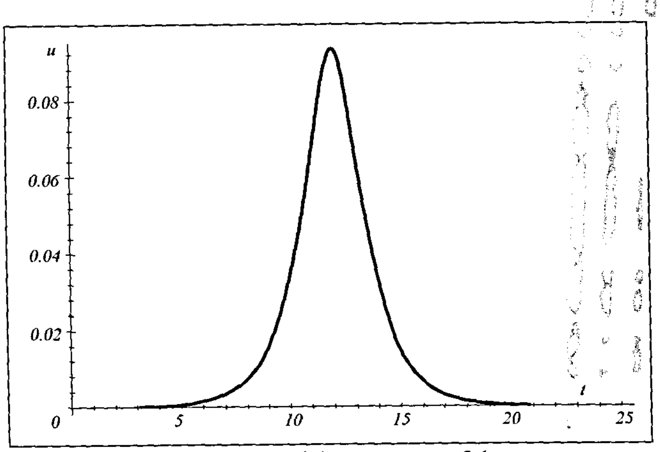

Fig. 7- Time history of the response of the system, disturbed from saddle point equilibrium

ESCAPE ENERGY METHOD

We are looking for another alternating method to evaluate u_m. If we write the equation of motion (18) in the following form:

$$u'' + u = u + f(u) \tag{30}$$

then, it is seen that

$$\frac{1}{2}E' = \frac{1}{2}\frac{d}{dz}\left(u'^2 + u^2\right) = [u + f(u)]u' \tag{31}$$

therefore,

$$E = 2\int [u + f(u)]u'dz \tag{32}$$

where E is called moving energy, due to variable $z=x+vt$. Figure 8 shows that the energy required to escape the state point of the system from second quarter space of phase space, $(u>0,\ u'<0)$, will be minimum for the saddle point trajectory.

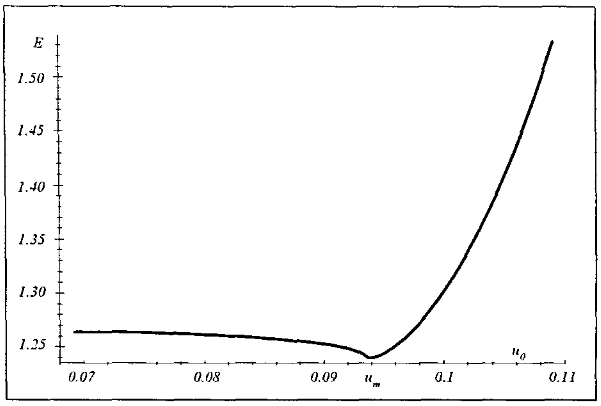

Fig 8- escape moving energy E versus u_0

ESCAPE TIME METHOD

Now we can evaluate the moving time of motion as

$$z = \int \frac{du}{\sqrt{2\int f(u)du}} \tag{33}$$

with the following initial conditions for equation (16),

$$u(0) = u_0 > u_i \quad , \quad u'(0) = 0 \tag{34}$$

where u_i is the position of the center equilibrium point, and is illustrated in figure 1. The point u_i is the position of inflection of the solitary wave illustrated in figure 5, as well.

The time required to escape the state point of the system from second quarter space of phase space, $(u>0,\ u'<0)$, will be maximum for the saddle point trajectory. The reason is the asymptotical approach of the saddle point trajectory to the equilibrium point. It is shown in figure 9, that the escape time Γ, is maximum at u_i.

$$\Gamma = \int_{u_m}^{u=0 \text{ or } u'=0} \frac{du}{\sqrt{2\int f(u)du}} \tag{35}$$

This method is applicable to any system for which the origin $(0,\ 0)$ is a saddle point equilibrium. Otherwise, the origin must be moved to the saddle point.

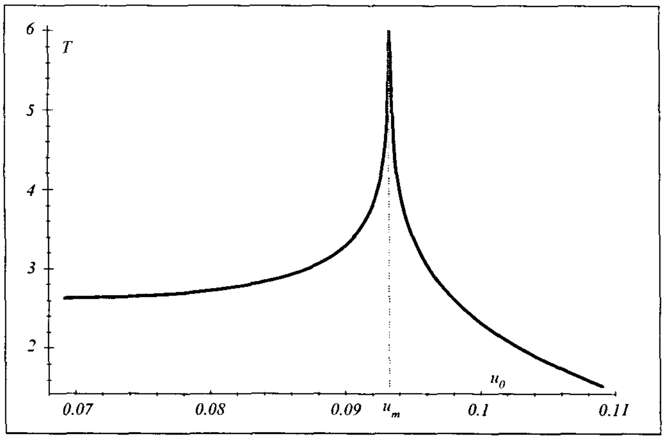

Fig. 9- escape moving time Z versus u_0

It is known that the behavior of the equation of motion (18) depends on the parameter v. The position of the equilibrium point u_i, is illustrated in figure 10 and dependence of amplitude of the solitary waves u_m, to v is illustrated in figure 11.

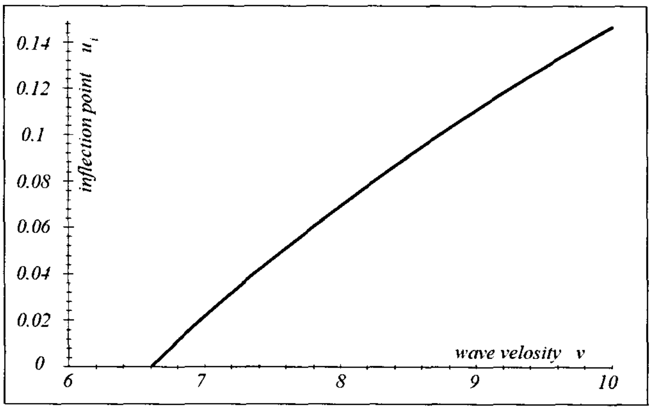

Fig. 10- position of the inflection point u_i

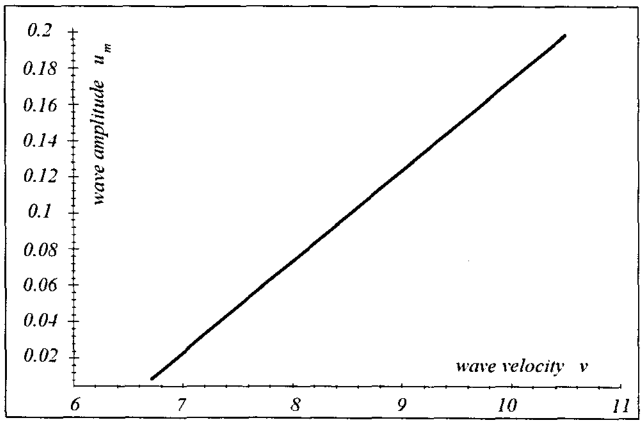

Fig. 11- dependence of the amplitude u_m to v

It can be shown that the position of the equilibrium point will appeared for those values of v, less than a critical value $v_c=6.66$. In fact, there is no solitary wave for $v < v_c$.

MODEL ANALYSIS

The equation of motion (18) is completed whenever the explicit form of the stress components s_z and s_θ are derived. For a symmetrically deformed membrane, the components of the principal stresses may be given as:

$$\sigma_z = \Lambda_z \frac{\partial \Sigma}{\partial \Lambda_z}, \quad \sigma_\theta = \Lambda_\theta \frac{\partial \Sigma}{\partial \Lambda_\theta} \tag{36}$$

where

$$\sigma_z = \mu\, s_z, \quad \sigma_\theta = \mu\, s_\theta \tag{37}$$

and

$$\Sigma = \Sigma(\Lambda_z, \Lambda_\theta) = \Sigma(\lambda_z, u) \tag{38}$$

Σ is the strain energy density function of the tube material. It is seen that s_z and s_θ are functions of $u(z, t)$, but the form of their functions depend on the mathematical model of strain energy density function, Σ. Equations (20) and (21) are derived from Demiray's model D1,

$$\Sigma_{D1} = \frac{\mu}{2\alpha}\left(e^{\alpha(I_1-3)} - 1\right) \tag{39}$$

It is known that the solitary wave has one bump and then exponentially tends to zero. Therefore, outside of a small interval where the pulse exists, the function goes to zero. The phase plane analysis showed that the function $f(u)$ must have a double zero at $u=0$ and another at $u=u_i>0$, and must be C^1 in this interval. It must be positive in the interval $(0, u_i)$ and negative in the interval (u_i, u_m). The separaterix orbit, which originates at $u=0$, surrounds the other equilibrium point u_i, and separates the domain of closed orbits around u_i. In addition, the integral of $f(u)$ between $u=0$ and $u=u_m$ must vanish.

If we rewrite the equation of motion (16), in the following form

$$u'' = \frac{m}{\Lambda_z \dfrac{\partial \Sigma}{\partial \Lambda_z} - \mu m v^2}\left(\frac{\Lambda_\theta \dfrac{\partial \Sigma}{\partial \Lambda_\theta}}{1+u} - \frac{1}{2}(p_0 + p)(1+u)\right) = f(u) \tag{41}$$

and enforce the mentioned condition $f(0)=0$ and $f(u_i)=0$, the following conditions on the function Σ are achieved:

$$\left[\Lambda_\theta \frac{\partial \Sigma}{\partial \Lambda_\theta}\right]_{u=0} = \frac{\mu}{2} p_0 \tag{42}$$

$$\left[\Lambda_\theta \frac{\partial \Sigma}{\partial \Lambda_\theta}\right]_{u=u_i} - (1+u_i)^2 \left[\Lambda_\theta \frac{\partial \Sigma}{\partial \Lambda_\theta}\right]_{u=0} = \frac{\mu}{2} p_i (1+u_i)^2 \tag{43}$$

where

$$p_i = \frac{v^2}{2}\left(1 - \frac{1}{(1+u_i)^4}\right) \tag{44}$$

The continuity condition shows that

$$\left[\Lambda_z \frac{\partial \Sigma}{\partial \Lambda_z}\right] \neq m v^2 \quad , \quad 0 < u < u_m \tag{45}$$

and in addition,

$$\left[\frac{\partial f}{\partial u}\right]_{u=0} > 0 \quad , \quad \left[\frac{\partial f}{\partial u}\right]_{u=u_i} < 0 \tag{46}$$

the integral condition leads to the following equation:

$$\int_{u=0}^{u=u_m} f(u,u')\,du = 0 \tag{47}$$

Now if one examines the following three energy functions, presented by Demiray, [12],

$$\Sigma_{D1} = \frac{\mu}{2\alpha}\left(e^{\alpha(I_1-3)} - 1\right) \tag{48}$$

Ishiara, [13],

$$\Sigma_I = \frac{\mu}{2}\left[\beta(I_1 - 3) + (1-\beta)(I_2 - 3) + \delta(I_1 - 3)^2\right] \tag{49}$$

and Demiray, [14],

$$\Sigma_{D2} = \frac{\mu}{2\alpha}\left(e^{v(I_1-3)} - 1\right) \tag{50}$$

one will find that Σ_{D1}, Σ_{D2}, satisfy all the conditions, and Σ_I could only be a satisfactory energy function for some β and δ.

Now, suppose that the function $f(u)$ be a given function, say

$$u'' = f(u) = -A(v)u^2 + B(v)u \tag{51}$$

$$\forall v: \ A(v) > 0, \ B(v) > 0 \tag{52}$$

This function satisfies the conditions (44) and (45), provided that

$$u_m = \frac{3B(v)}{A(v)} \tag{53}$$

Substitution (49) in (39), reduces to the following parametric partial differential equation for determining the energy function Σ.

$$K(\Lambda_z,\Lambda_0)\frac{\partial\Sigma}{\partial\Lambda_z}-G(\Lambda_z,\Lambda_0)\frac{\partial\Sigma}{\partial\Lambda_0}=H(\Lambda_z,\Lambda_0) \tag{54}$$

It is known that there exists no general method for solving this differential equation. Thus one must consider physical features in addition to the mathematical considerations, in order to find a satisfactory energy function.

CONCLUSION

One of the principal characteristics of solitary waves is their amplitude. Although there is no general analytical method, the amplitude can be found by some numerical and non-straightforward methods.

The style of solitary waves depends on the given model for strain energy density function. There are some necessary physical conditions, which must be satisfied by the strain energy function.

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