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STABILITY ANALYSIS OF A LINEARIZED MEMS

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ABSTRACT
This paper presents the stability theory and dynamic behavior of a micro-mechanical parametric-effect resonator. The device is a MEMS time-varying capacitor. The nonlinear dynamics of the MEMS are investigated analytically, and numerically. Applying perturbation methods, and deriving an analytical equation to describe the frequency response of the system enables the designer to study the effect of changes in the system parameters that can be used for design and optimization of the system.

INTRODUCTION
The micro-machining field, known as micro-electro-mechanical systems (MEMS), is microscopic mechanical systems coupled with electronic circuits. Every MEMS generally has an input transducer, a mechanical resonator and an output transducer. In this investigation, a MEMS with variable capacitor is studied.

Mathematical model of micro-mechanical resonator indicates a nonlinear parametric system. Their governing equation is parametric and hence, the stability of the system depends on the value of its parameters. From a design viewpoint, a stability chart is needed to indicate the relationship between the parameters to determine when the system is stable, periodic, or unstable. MEMS is destined to become a hallmark 21st-century manufacturing technology with numerous and diverse applications, having a dramatic impact technology. Examples of likely MEMS applications are medical instrumentation for in-body surgery, hearing aids, air-bag sensors, micro pumps and optics and tilting mirrors for projection devices [1]. As this breakthrough technology allows unparalleled synergy between hitherto unrelated fields of endeavor, MEMS is forecast to have a commercial and defense market growth rate similar to that of its parent IC technology [2].

In this paper, a model of MEMS, shown in Figure 1, will be analyzed. The governing equation of the MEMS is a nonlinear parametric equation. Using perturbation methods and Energy-Rate method [3], the stability of the system indicating boundaries of stable and unstable regions is studied.

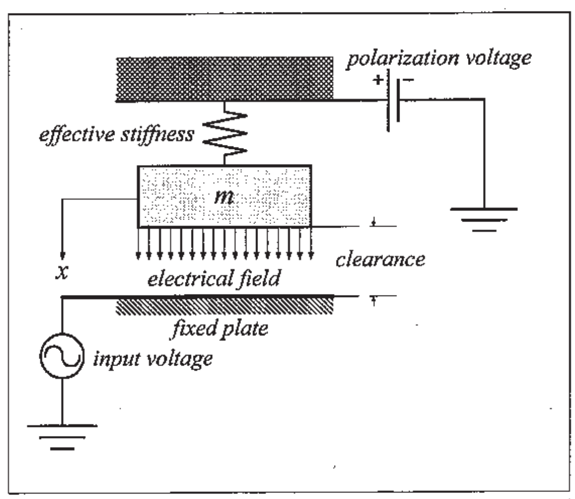

Figure1. A simplified time-varying mechanical model of the MEMS

MODELING

A model for the MEMS is shown in Figure 1. The fixed plate of the capacitor with area A, connected to an alternating current voltage $v_i = v \sin(\omega t)$. v_i, and ω, are the AC amplitude, and frequency respectively. The moving plate of the capacitor is a plate with mass m. The supporting suspension of the moving plate is simplified as a nonlinear spring of stiffness $k = k_1 + k_2 x^2$, parallel to a linear damper of damping c. The moving plate, might be connected to a polarization voltage v_p.

The coordinate used to measure the displacement of the moving plate is x, and the equation of motion for the mechanical resonator in the MEMS would be:

$$m\ddot{x} + c\dot{x} + kx = f \tag{1}$$

where, the electric force f is

$$f = \frac{\varepsilon_0 A (v - v_p)^2}{2(d-x)^2} = \frac{\varepsilon_0 A}{2(d-x)^2} \times \left[v_P^2 + \frac{1}{2} v_i^2 + 2 v_P v_i \sin(\omega t) - \frac{1}{2} v_i^2 \cos(2\omega t) \right] - \frac{\varepsilon_0 A v_P^2}{2d^2} \tag{2}$$

ε_0 as the permittivity in vacuum, d is the gap size, and A is the area of the plate. A more detailed calculation of m, c, and k are presented in [1]. Introducing a set of variables, we may transform the equation of motion to the following form

$$y'' + hy' + y + \lambda y^3 = \frac{1}{(1-y)^2} \times \left[(\alpha + \beta) + 2\sqrt{2\alpha\beta} \sin(r\tau) - \beta \cos(2r\tau) \right] - \alpha \tag{3}$$

where

$$\tau = \omega_n t \qquad \omega_n = \sqrt{\frac{k_1}{m}} \qquad y = \frac{x}{d} \qquad r = \frac{\omega}{\omega_n}$$

$$h = \frac{c}{\sqrt{k_1 m}} \qquad \alpha = \frac{\varepsilon_0 A}{2 k_1 d^3} v_P^2 \qquad \beta = \frac{\varepsilon_0 A}{4 k_1 d^3} v_i^2 . \tag{4}$$

$$2\sqrt{2\alpha\beta} = \frac{\varepsilon_0 A}{k_1 d^3} v_P v_i \qquad \lambda = \frac{k_2}{k_1} d^2$$

Polarization voltage changes the static position of m. Assuming a linear spring, $k = k_1$ the equilibrium position of the system is at the roots of $y(1-y)^2 = \alpha$.

Assuming no polarization voltage, simplifies the electromagnetic force and reduces the equation of motion to

$$y'' + hy' + y + \lambda y^3 = \frac{1}{(1-y)^2} \left[\beta - \beta \cos(2r\tau) \right] \tag{5}$$

The stable point $y=0$ is the only equilibrium point of the system (5) in absence of alternative voltage. Therefore, its dynamic is much simpler. No polarization model of the MEMS is what Napoli et. al. [4] have used to study the parametric resonance of the system.
The investigation method is to create and apply a reliable nondimensionalized mathematical model of the MEMS that will provide key dynamic properties of the system.

ANALYSIS OF LINEAR MODEL WITH POLARIZATION VOLTAGE MODEL

Series expansion of the nonlinear part of Equation (3) indicates that:

$$\frac{1}{(1-y)^2} = 1 + 2y + 3y^2 + 4y^3 + O(y^4) . \tag{6}$$

Assuming $y \ll 1$ converts the equation of motion to

$$y'' + hy' + \left(1 - 2\beta - 2\alpha + 2\beta \cos(2r\tau) - 4\sqrt{2\alpha\beta} \sin(2r\tau)\right) y = 2\beta \sin^2(r\tau) + 2\sqrt{2\alpha\beta} \sin(r\tau) \tag{7}$$

It can be shown that applying averaging method provides the following frequency response.

$$4\beta^2 Y^4 r^4 + 4\beta^2 Y^4 r^2 \left(4(\beta+\alpha) + h^2 - 2\right) + 2\beta^2 Y^4 (1 - \beta - 2\alpha)(1 - 3\beta - 2\alpha) - 16\beta^3 Y^2 \left[\sqrt{\alpha(\alpha - r^2 Y^2) + \alpha(1 - \beta - 2\alpha) Y^2} - \alpha \right] = 0 \tag{8}$$

A plot of Equation (8), shown in Figure 2, indicates that polarization voltage make the system vibrate as a well behaved resonator. Since nonlinearities are ignored, there would be no jump in steady state response of the MEMS. The frequency response is stable in this case. The effect of varying parameters β and α are plotted in Figures 3 and 4 respectively.

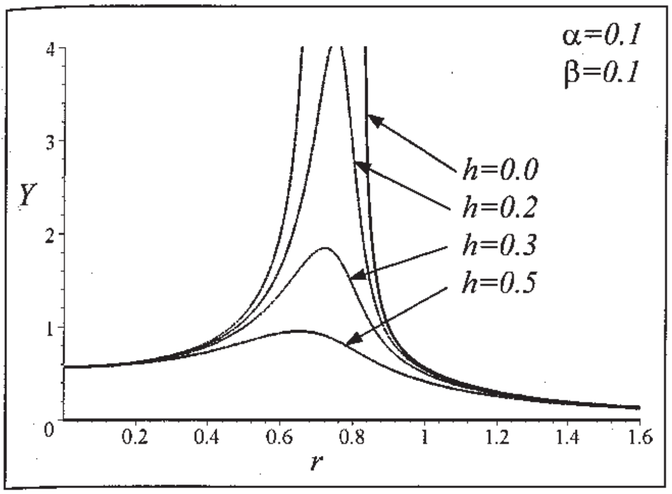

Figure 2. Frequency response of the linearized MEMS with polarization for a set of parameters

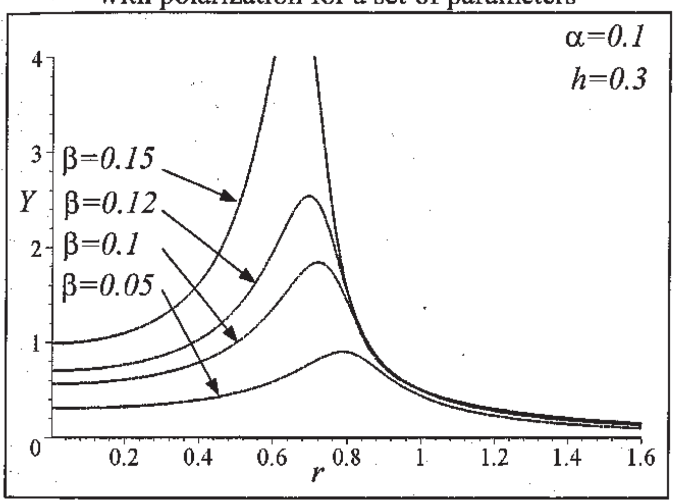

Figure 3. Frequency response of the linearized MEMS with polarization for a set of parameters

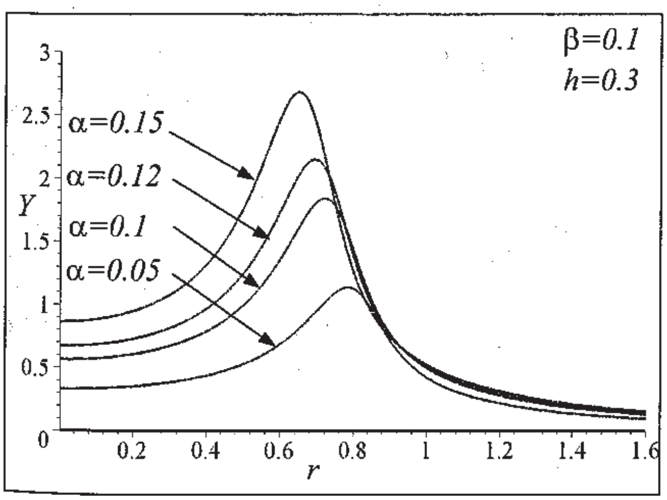

Figure 4. Frequency response of the linearized MEMS with polarization for a set of parameters

ANALYSIS OF LINEAR AND WITH NO POLARIZATION VOLTAGE MODEL

Series expansion of the nonlinear part of Equation (3) indicates that:

$$\frac{1}{(1-y)^2}=1+2y+3y^2+4y^3+O(y^4)\,. \tag{9}$$

Assuming $y \ll 1$ converts the equation of motion to a forced Mathieu differential equation

$$\begin{aligned} &y''+hy'+\left(1-2\beta+2\beta\cos(2r\tau)\right)y \\ &=2\beta\sin^2(r\tau) \end{aligned} \tag{10}$$

The transition curve in the stability plane r-β after applying averaging method is

$$\begin{aligned} &2r^2=2-h^2-4\beta \\ &\pm\sqrt{h^4-4h^2(1-2\beta)+4\beta^2} \end{aligned}\,. \tag{11}$$

Since $r^2 \in \mathbf{R}$, then h must be within

$$\begin{aligned} &2-4\beta-2\sqrt{3\beta^2+1-4\beta}<h^2 \\ &h^2<2-4\beta+2\sqrt{3\beta^2+1-4\beta} \end{aligned},$$

to have a transition curve. The transition curves and boundary of stability for the first instability tongue are plotted in Figure 5 for different value of h. As expected, the instability domain shrinks by increasing damping.

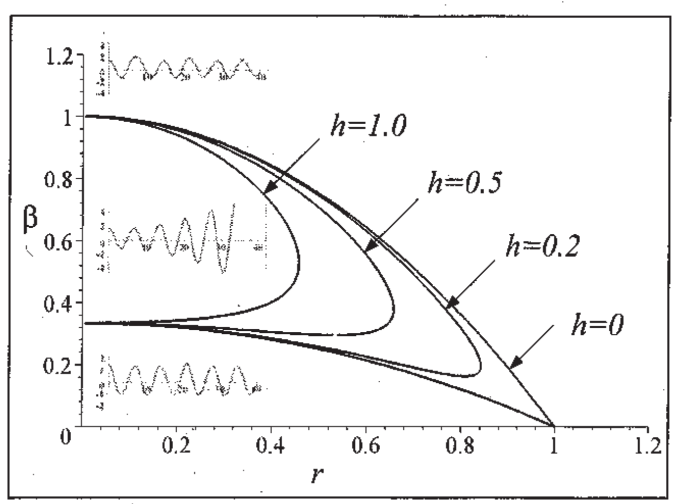

Figure 5. Transient curves and instability tongue for linear and no polarization model of the MEMS.

Applying Poincare-Lindstad method provides the following equations for no damping transition curves

$$\beta_1 = 1 - r^2 + O(r^4)$$
$$\beta_2 = \frac{1}{3} - \frac{r^2}{3} + O(r^4) \qquad (12)$$

which are in agreement with Equation (11) and Figure 6. When the transient curves in parametric plane β-*r* are determined by averaging or Poincare-Lindstad methods, the stable and unstable regions can be determined by investigating time response of a picked point. Selecting $(r,\beta) = (0.2, 0.1)$, *(0.9, 0.1)*, and *(1.2, 0.1)* in the three regions divided by curves given in (12) for *h=0*, generates the time responses shown in Figure 5.

Although Equations (11) and (12) determine the periodic curves in the β-*r* parameter space around β=*0*, and *r=1*, perturbation methods cannot determine a global stability regions. Applying Energy-Rate method provides an exact stability diagram shown in Figure 5, [4]. In Figure 6, the parameter β is plotted against *1/r* to provide a better view for *r<1*.

As can be seen in Figure 6, the linear no-polarization model of the MEMS has a complicated stability diagram, although there might be some limits for acceptable domain of β and *r* due to physical restrictions.

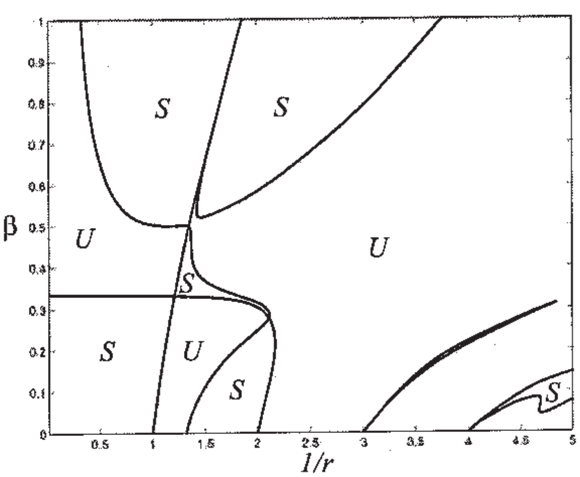

Figure 6. Stability diagram for linear and no polarization model of the MEMS

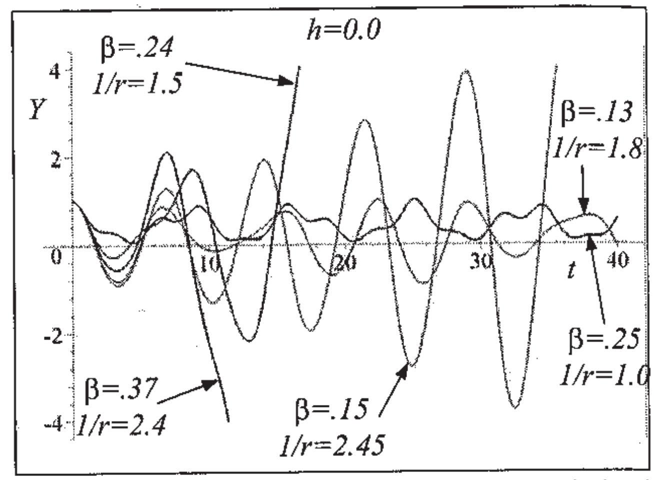

Figure 7. Time response of the linear and no-polarization model of the MEMS for some points of the parameter plane

The stability characteristic of regions in Figure 6 is obvious by investigating time responses (curves) in Figure 7. It can also be determined by direct numerical simulation. The principal instability region connected to *1/r=1* is more important, and is what the perturbation method could predict approximately.

REFERENCES

[1] J. P. Raskin, A. R. Brown, B. T. Khuri-Yakub, and G. M. Rebeiz, "A novel parametric effect MEMS amplifier", *Journal of Microelectromechanical Systems, 2000, 9(4), 528-537.*

[2] J. L. A. Yeh, and C. Y. Hui, "Electrostatic model for an asymmetric comb drive", *Journal of Micro-electromechanical Systems, 2000, 9(1), 126-135.*

[3] G. Nakhaie Jazar, "Stability Chart of Parametric Vibrating Systems, Using Energy Method", *International Journal of Nonlinear Mechanics, 2004, 39(8), 1319-1331.*

[4] M. Napoli, R. Baskaran, K. Turner, and B. Bamieh, "Understanding Mechanical Domain Parametric Resonance in Microcantilevers", *IEEE The Sixteenth Annual International Conference on Micro Electro Mechanical Systems, MEMS-03 Kyoto, 19-23 Jan. 2003.*