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VEHICLES AND NONLINEAR SUSPENSIONS

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ABSTRACT

An independent suspension for conventional vehicles has been modeled as a nonlinear vibration absorber with a nonlinear third-order ordinary differential equation. In order to obtain conditions that guarantee existence of periodic solutions and stable responses, the Schauder's fixed-point theorem has been implemented to prove a third-order solution existence theorem for general third-order differential equations.

A numerical method has been developed for rapid convergence, and applied for a sample model. The correctness of sufficient conditions and solution algorithm has been shown with appropriate figures.

INTRODUCTION

One of the most important characteristics and problems of car suspension is that only a fixed and limited suspension working space is available, and that such vehicles have to traverse on road surfaces of widely differing roughness. These results make it clear that the chief limitation of conventional fixed parameter passive suspension systems arise from the need of compromise in choice of parameters. They must be chosen according to the opposite demands of smooth and rough surfaces, vehicle attitude and load changes, and maneuvering and high speed handling quality (Sharp and Hassan 1986).

In order to facilitate such compromise, the relationship between the extent of the stiffness and damping variations must be provided and the performance gains must be obtained (Pacejka 1986). Using nonlinear springs might be a way to overcome some of the limitations. With nonlinear elements, the simplicity and stability of the system will change and therefore, its behavior should be analyzed.

The assumption of linear behavior of mechanical elements simplifies the solution considerably but it is too ideal for most real systems. Nonlinear dynamical systems, which are a more realistic representation of nature, could exhibit a somewhat complex behavior. Their analysis requires a thorough investigation of the solution of the nonlinear governing differential equations, which usually do not provide any exact solutions and therefore must be solved numerically.

Perhaps, the most important problem in the study of nonlinear dynamical systems is to obtain conditions that guarantee the existence of periodic solutions, and hence calculate these solutions by implementing suitable numerical techniques. The significance of periodic solutions lies on the fact that these solutions represent the steady state response of the system.

We have modeled the front suspension system of conventional vehicles as nonlinear dynamic vibration absorber systems using a third-order ordinary differential equation. Then, we have attempted to obtain sufficient conditions for periodicity of the corresponding system responses using Schauder's fixed-point theorem. After obtaining the periodicity conditions, we have solved the system numerically. The obtained numerical solutions not only demonstrate the response of the system, but also offer a way to check whether the proposed sufficient periodicity conditions are valid.

VIBRATION OF VEHICLES WITH NONLINEAR SUSPENSION

Figure 1 represents the essential parts of the front suspension of a motorcar. The unsprung mass consists of the tire, the wheel, and the stub axle, and is connected to a hydraulic shock absorber and the main spring by a rubber bushing. The other end of the shock absorber is connected to the sub-frame of the car body by another rubber bushing. One wishbone arm at each end serves to stabilize the unit.

This mechanical system can be considered as a two degree-of-freedom dynamical system. The tire stiffness is assumed to be large enough compared to that of the main spring, and hence the suspension system can be simplified in the form of a single degree-of-freedom system (Esmailzadeh 1978).

Let x, y and z represent the body motion, wheel excitation, and the displacement at the connection point of the rubber bushing and the hydraulic shock absorber respectively as shown in Figure 2. Then, the equations of motion may be written as:

$$-k_1(x-y)-k_2(x-z)=mx'' \tag{1}$$

$$k_2(x-z)=c(z'-y') \tag{2}$$

In order to obtain the relation between the input displacement y and the output motion x, the variable z should be eliminated between equations (1) and (2). Thus,

$$z=\frac{\left[mx''+(k_1+k_2)x-k_1y)\right]}{k_2} \tag{3}$$

Substituting equation (3) in (2) we obtain,

$$x'''+\frac{k_2}{c}x''+\frac{k_1+k_2}{m}x'+\frac{k_1k_2}{mc}x=\frac{k_1+k_2}{m}y'+\frac{k_1k_2}{mc}y \tag{4}$$

If a periodic profile is assumed for the road and the vehicle is traveling at a constant speed, the input displacement y can be well represented by a periodic function.

In the case of linear springs and dampers, the solution of this third-order differential equation could be easily obtained. However, due to nonlinear behavior of real mechanical springs and shock absorbers, such an over-simplification is not always realistic. For the proposed model with nonlinear elements, a third-order nonlinear differential equation is obtained.

Experiments show that with relatively large displacements, the spring rate may be expressed as: $a+b\delta^2$, where a is a positive constant. For a hard spring, the constant factor b is also positive. On the other hand, for the case of a soft spring b ought to be negative. The factor δ represents the relative displacement of the two ends of the considered spring. Hence, for the main spring, $\delta=x-y$.

Let's consider both k_1 and k_2 be nonlinear functions of x, and c be a function of x'. For this special case, equation (4) will have the following general form,

$$x'''+g_1(x')x''+g_2(x)x'+g(x,x',t)=e(t) \tag{5}$$

where

$$g_1(x')=\frac{k_2}{c} \qquad g_2(x)=\frac{k_1+k_2}{M}$$
$$g(x,x',t)-e(t)=\frac{k_1k_2}{mc}x-\frac{k_1+k_2}{m}y'-\frac{k_1k_2}{mc}y \tag{6}$$

As an example, we study a system with all elements being linear except the main spring k_1. Hence, assuming $k_1=a+b\delta^2$, $\delta=x-y$, one may write equation (4) as:

$$\begin{aligned} &x'''+\frac{k_2}{c}x''+\frac{a+k_2}{m}x'+\frac{b}{mc}(x-y)^2x' \\ &\quad+\frac{ak_2}{mc}x+\frac{bk_2}{mc}(x-y)^2x= \\ &\frac{a+k_2}{m}y'+\frac{b}{m}(x-y)^2y'+\frac{ak_2}{mc}y+\frac{bk_2}{mc}(x-y)^2y \end{aligned} \tag{7}$$

MATHEMATICAL ANALYSIS

Consider the following classes of nonlinear differential equations:

$$x''+g_1(x)x'+g(x,x',t)=e(t) \tag{8}$$

$$x'''+g_1(x')x''+g_2(x)x'+g(x,x',t)=e(t) \tag{9}$$

Equation (8) is a second-order nonlinear differential equation, whose periodic solutions are discussed (Mehri, Esmailzadeh, and Nakhaie Jazar 1996). Equation (9) is a third-order nonlinear differential equation that is an extension to Equation (8). In general case, the exact solution of equation (9) is not known. Hence, various numerical techniques should be utilized to determine its approximate periodic solutions. However, the Schauder's fixed-point theorem enables us to find the existence conditions of periodic solutions, without evaluating such answers. It is interesting to note that the conditions that ensure the existence of periodic solutions for equation (8) are also valid for equation (9). This fact is discussed later.

Let us consider equation (9). The aim is to obtain conditions for the periodicity of solution that has the same time period as that of the input excitation. The method presented here is based on the Schauder's fixed-point theorem (Nakhaie Jazar and Golnaraghi 2002).

Assume that g and e are periodic functions of t. The necessary and sufficient condition for equation (9) to have a periodic solution x with the same time period τ as e is:

$$x^{(i)}(0) = x^{(i)}(\tau) \qquad i = 0,1,2 \tag{10}$$

where *(i)* indicates the *i*-th time derivative. If we introduce the Green's function *G(t,s)*, the solution of equation (9) can be expressed as:

$$x(t) = \int_0^\tau G(t,s)\big[g_1(x'(s))x''(s) + g_2(x(s))x'(s) + g(x(s),x'(s),s) - e(s)\big]ds \tag{11}$$

Assume that the forcing function has the following property:

$$\int_0^\tau e(t)dt = 0 \tag{12}$$

Now, by combining equation (12) with equation (11), one can state that the periodicity condition (10) can be satisfied, if *g* satisfies the following condition:

$$\int_0^\tau g(x_0'(s),x'(s),s)ds = 0 \tag{13}$$

where $x_0(t)$ is the periodic solution to equation (9). Equation (13) expresses the sufficient condition for periodicity of the solution to equation (9). In order to find conditions that ensure the existence of $x_0(t)$ which satisfies equations (11) and (13), the Schauder's fixed-point theorem may be applied.

Let $C[0,\tau]$ be the space of all differentiable functions on $[0,\tau]$ equipped with the following norm:

$$\|x\| = Max\{|x(t)|; \quad t \in [0,\tau]\} \tag{14}$$

The complete normed linear space *B* can be defined in the following form:

$$B = C[0,\tau] \times C[0,\tau] \times C[0,\tau] \times R \tag{15}$$

The norm of *B* elements can be defined as:

$$\|(x,x',x'',h)\| = |x| + |x'| + |x''| + |h| \tag{16}$$

On the space *B*, the operator *U* can be defined as following:

$$U(x,x',x'',h) = (\bar{x},\bar{x}',\bar{x}'',\bar{h}) \tag{17}$$

where

$$\bar{x}^{(i)}(t) = h^{(i)} + +\int_0^\tau G^{(i)}(t,s)\big[g_1(x'(s))x''(s) + g_2(x(s))x'(s) + g(x(s),x'(s),s) - e(s)\big]ds \tag{18}$$

$$\bar{h} = h - \frac{1}{\tau}\int_0^\tau g(\bar{x}(s),\bar{x}'(s),s)ds \tag{19}$$

Hence, the operator *U* represents a continuous mapping from *B* into itself. A closed convex subset of *B* can be defined as:

$$S = \{(x,x',x'',h) \in B; \; |x| \le K + |x'| \le K + |x''| \le K + |h| \le (v + 2m)\} \tag{20}$$

where

$$|x| \ge v; \quad v \ge 0; \quad t \in [0,\tau] \tag{21}$$

$$N = Max\{MM_0\tau, \quad MM_1\tau, \quad MM_2\tau, \quad F\} \tag{22}$$

$$F = Max\{|g(x,x',t)|; t \in [0,\tau], |x| \le K\} \tag{23}$$

$$M = Max\{|g_1(x'(t))x''(t) + g_2(x(t))x'(t) + g(x(t),x'(t),t) - e(t)|; \; t \in [0,\tau], |x^{(i)}| \le K, i = 0,1,2\} \tag{24}$$

$$M_i = Max\left\{\left|\frac{\partial^i G(t,s)}{\partial t^i}\right|; (t,s) \in [0,\tau] \times [0,\tau]\right\} \quad i = 0,1,2 \tag{25}$$

If it is shown that the operator *U* has a fixed point in the set *S*, there is a function x_0 for which

$$U(x_0,x_0',x_0'',h_0) = (x_0,x_0',x_0'',h_0) \tag{26}$$

Considering equations (18) and (19), one can see that x_0 must then satisfy both equations (11) and (13). Consequently, x_0 will be the desired periodic solution of equation (9).
According to the Schauder's fixed-point theorem, existence of a fixed point is proved if:

$$U(S) \subset S \tag{27}$$

It can be shown that if $v + 3N < K$, for any

$$(x,x',x'',h) \in S \tag{28}$$

its corresponding transformation, $(\bar{x},\bar{x}',\bar{x}'',\bar{h})$ is also a member of *S* and the proof is completed. Regarding the foregoing discussion, a theorem can be deduced:

Third Order Periodic Theorem: For a differential equation in the form of equation (9) with periodicity conditions given by equation (10), there exists at least one solution with the same time period τ as that of functions *g* and *e* provided that:

$$v + 3N \le K ; \quad xg(x, x', t) > 0 , \quad t \in [0, \tau] \tag{29}$$

PERIODICITY CONDITION FOR VIBRATION OF VEHICLE SUSPENSION

Assume that the input excitation of the system with equation (7) is $y = y_0 \cos(2\pi t)$. Then the sufficient condition (23) for periodicity of the response of the system with $v = 0$ is

$$\begin{aligned} &\frac{k_2}{c}K + \frac{a+k_2}{m}K + \frac{b}{m}K(K+y_0)^2 + \frac{ak_2}{mc}K + \\ &+\frac{a+k_2}{m}(2\pi y_0) + \frac{b}{m}(K-y_0)^2(2\pi y_0) + \\ &+\frac{ak_2}{mc}y_0 + \frac{bk_2}{mc}(K-y_0)^2 y_0 < \frac{K}{3} \end{aligned} \tag{30}$$

Consider the following numerical values:

$$\begin{aligned} k_2 &= 100 \ \ kgf/cm \\ c &= 1000 \ \ kgf \cdot s/cm \\ a &= 500 \ \ kgf/cm \\ m &= 5000 \ \ kg \\ b &= 10 \ \ kgf/cm^3 \\ y_0 &= 0.1 \ \ cm \end{aligned} \tag{31}$$

it may be verified that inequality (29) is being satisfied for $K=0.76$.

NUMERICAL PROCEDURE

Now, we are certain that a periodic solution exists. The next step is to calculate this solution by use of numerical methods. The differential equation is assumed to have the form

$$x''' + g(x, x', x'', t) = e(t) \tag{32}$$

in which all explicit functions of time are assumed to be periodic with period τ. The purpose of the present discussion is to calculate solutions of equation (32) that are periodic with the same period τ. Hence, such a solution should satisfy the boundary conditions (10).

Regarding the foregoing explanations, the problem reduces to finding proper values for

$$\alpha = x(0) \quad \beta = x'(0) \quad \gamma = x''(0) \tag{33}$$

such that the corresponding solution of equation (33) satisfies the following set of algebraic equations:

$$\begin{aligned} \varphi(\alpha,\beta,\gamma) &= x(\alpha,\beta,\gamma,\tau) - \alpha = 0 \\ \theta(\alpha,\beta,\gamma) &= x'(\alpha,\beta,\gamma,\tau) - \beta = 0 \\ \psi(\alpha,\beta,\gamma) &= x''(\alpha,\beta,\gamma,\tau) - \gamma = 0 \end{aligned} \tag{34}$$

Now, one should guess $(\alpha_0,\beta_0,\gamma_0)$ as the initial conditions. Then equation (33) can be solved numerically to evaluate $x(\tau)$, $x'(\tau)$, $x''(\tau)$. The validity of the initial guess may be checked with the following criterion:

$$\left|\frac{\varphi(\alpha_0,\beta_0,\gamma_0)}{\alpha_0}\right| + \left|\frac{\theta(\alpha_0,\beta_0,\gamma_0)}{\beta_0}\right| + \left|\frac{\psi(\alpha_0,\beta_0,\gamma_0)}{\gamma_0}\right| < \varepsilon \tag{35}$$

where ε is a convergence tolerance. If it is valid, then $(\alpha_0, \beta_0, \gamma_0)$ will be the proper set of initial conditions and the corresponding solution of equation (33) will be τ-periodic. If it is not valid, the Newton-Raphson method can be applied to obtain a more feasible set of initial conditions $(\alpha_1, \beta_1, \gamma_1)$, such that

$$\alpha_1 = \alpha_0 + \Delta\alpha_1 , \quad \beta_1 = \beta_0 + \Delta\beta_1 , \quad \gamma_1 = \gamma_0 + \Delta\gamma_1 \tag{36}$$

Using Taylor series expansions and neglecting all second and higher order terms, one obtains

$$\begin{aligned} \varphi(\alpha_1,\beta_1,\gamma_1) &\approx \varphi(\alpha_0,\beta_0,\gamma_0) + \frac{\partial\varphi}{\partial\alpha}\Delta\alpha_1 + \frac{\partial\varphi}{\partial\beta}\Delta\beta_1 + \frac{\partial\varphi}{\partial\gamma}\Delta\gamma_1 \\ \theta(\alpha_1,\beta_1,\gamma_1) &\approx \theta(\alpha_0,\beta_0,\gamma_0) + \frac{\partial\theta}{\partial\alpha}\Delta\alpha_1 + \frac{\partial\theta}{\partial\beta}\Delta\beta_1 + \frac{\partial\theta}{\partial\gamma}\Delta\gamma_1 \\ \psi(\alpha_1,\beta_1,\gamma_1) &\approx \psi(\alpha_0,\beta_0,\gamma_0) + \frac{\partial\psi}{\partial\alpha}\Delta\alpha_1 + \frac{\partial\psi}{\partial\beta}\Delta\beta_1 + \frac{\partial\psi}{\partial\gamma}\Delta\gamma_1 \end{aligned} \tag{37}$$

where all derivatives are calculated at $(\alpha_0, \beta_0, \gamma_0)$. According to equations (35), the proper set of increments $\Delta\alpha_1$, $\Delta\beta_1$, $\Delta\gamma_1$, shall be calculated through solution of equations (37) with zero right hand sides. Finally, equation (37) is used to provide the improved initial conditions. This procedure should be repeated until the proper set of initial conditions, satisfying inequality (36), is obtained.

For calculating the derivatives in equation (37) effectively, one may find the following relations by using equations (35):

$$\begin{aligned} &\frac{\partial\varphi}{\partial\alpha} = \frac{\partial x(\alpha,\beta,\gamma)}{\partial\alpha} - 1 \quad && \frac{\partial\varphi}{\partial\beta} = \frac{\partial x}{\partial\beta} \quad && \frac{\partial\varphi}{\partial\gamma} = \frac{\partial x}{\partial\gamma} \\ &\frac{\partial\theta}{\partial\alpha} = \frac{\partial x'(\alpha,\beta,\gamma)}{\partial\alpha} \quad && \frac{\partial\theta}{\partial\beta} = \frac{\partial x'}{\partial\beta} - 1 \quad && \frac{\partial\theta}{\partial\gamma} = \frac{\partial x'}{\partial\gamma} \\ &\frac{\partial\psi}{\partial\alpha} = \frac{\partial x''(\alpha,\beta,\gamma)}{\partial\alpha} \quad && \frac{\partial\psi}{\partial\beta} = \frac{\partial x''}{\partial\beta} \quad && \frac{\partial\psi}{\partial\gamma} = \frac{\partial x''}{\partial\gamma} - 1 \end{aligned} \tag{38}$$

The partial derivatives of x, x' and x'' with respect to α, β or γ at any point are obtained by imposing perturbations on the corresponding initial conditions and then analyzing the effects of such perturbations in $x(\tau)$, $x'(\tau)$ and $x''(\tau)$.

RESULTS

Figure 3 depicts the periodicity condition (29). With a computer program based on the foregoing numerical techniques, we have detected the feasible initial conditions for periodicity. Then, the time histories of x, x' and x'' have been found and corresponding phase trajectory have been obtained. Figures 4-7 show the mentioned diagrams respectively. The parameters of the system have been chosen identical to the numerical values given in (32)

It is interesting to note that the x-x'' diagram is in the form of a line segment passing through the origin, which is swept at each time period. Hence, this solution seems to be a harmonic response.

Proper initial conditions for periodic answer were computed as:

$$\begin{aligned} x(0) &= -3.05928\times 10^{-4} \\ x'(0) &= 5.0073\times 10^{-6} \\ x''(0) &= 1.203694\times 10^{-2} \end{aligned} \tag{39}$$

for which:

$$\begin{aligned} x(0) &= -3.05928\times 10^{-4} \\ x'(0) &= 5.0091\times 10^{-6} \\ x''(0) &= 1.203694\times 10^{-2} \end{aligned} \tag{40}$$

These compare very well with conditions (10), and are in agreement with the previously computed value of K.

CONCLUSIONS

A vehicle suspension was modeled as a nonlinear vibration system. Its governing deferential equation was explained by a nonlinear third-order differential equation. Existence of periodic and therefore, stable responses which is not an obvious feature for nonlinear systems must be guaranteed. Thus the sufficient conditions for existence of periodic solutions for a general class of third-order ordinary differential equations, which includes the equation of mentioned suspension, were obtained. It was shown that these conditions could be applicable for analyzing the steady-state behavior of the suspension system. Therefore, the suspension could have a periodic response with a constant amplitude, when the excitation of the road is harmonic.

NOMENCLATURE

g_i, $i=0,1,2$	*coefficient functions*
$e(t)$	*forcing function*
C	*continuous and differentiable functions*
$x(t)$, $x'(t)$, $x''(t)$	*state variables*
$G(t,s)$	*Green's function*
S, B	*Banach spaces*
K	*bounded domain of phase space*
U	*operator*
M	*maximum of a function in phase space*
t	*time*
d, ∂	*differential symbols*
$'$ (prime)	*d/dt*
k_i, $i=1,2$	*stiffness coefficients of springs*
c	*damping coefficient of shock absorber*
m	*mass*
τ	*period*
φ, θ, ψ	*error functions*
α, β, γ	*initial conditions*
Δ	*increment in initial conditions*

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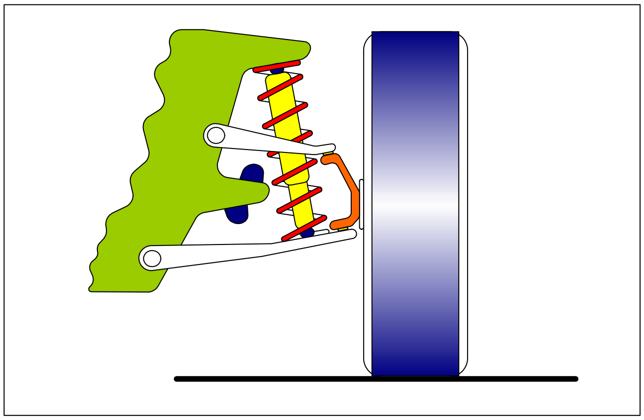

Figure 1. Main parts of a front suspension system

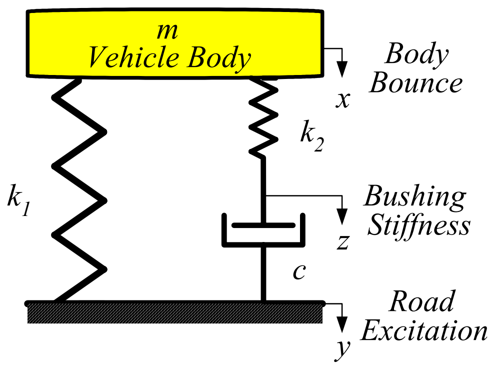

Figure 2. A model for vehicle suspension system

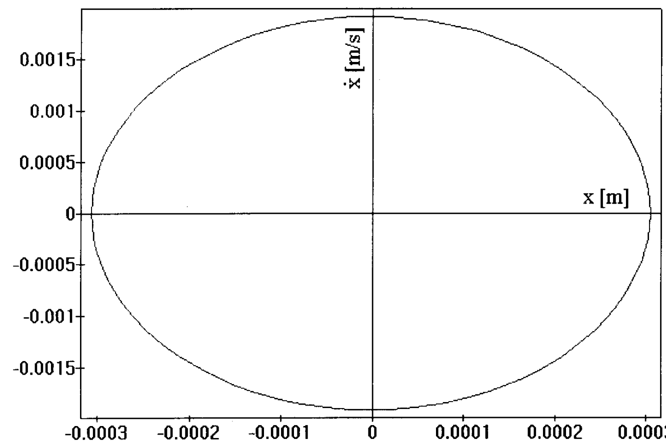

Figure 5. Phase plane trajectory for the periodic response of the system

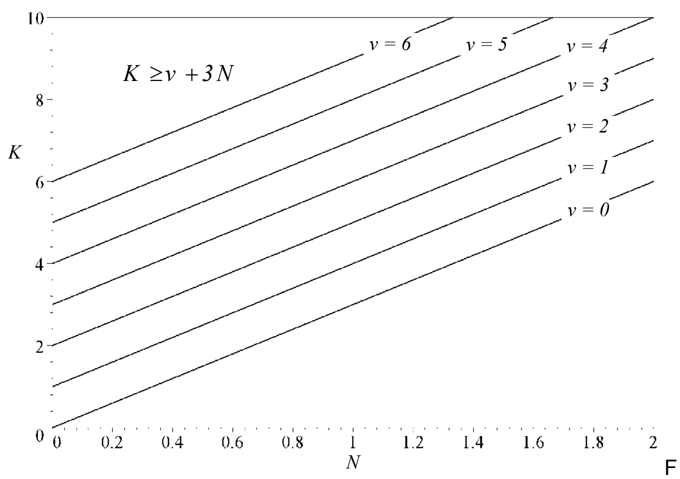

Figure 3. Graphical illustration of the periodicity condition

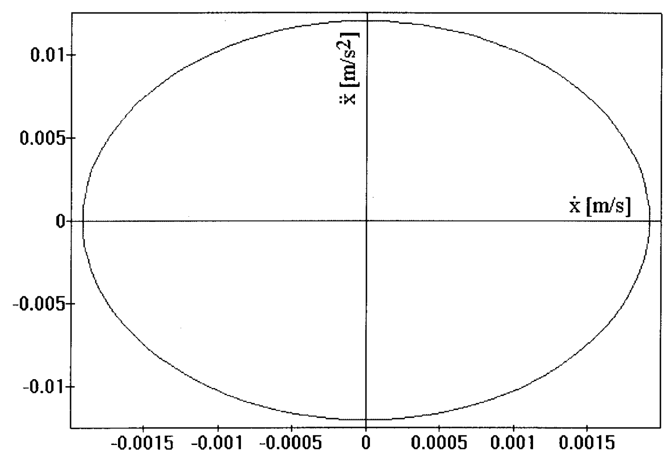

Figure 6. Plot of $\dot{x} - \ddot{x}$ for the periodic response of the system

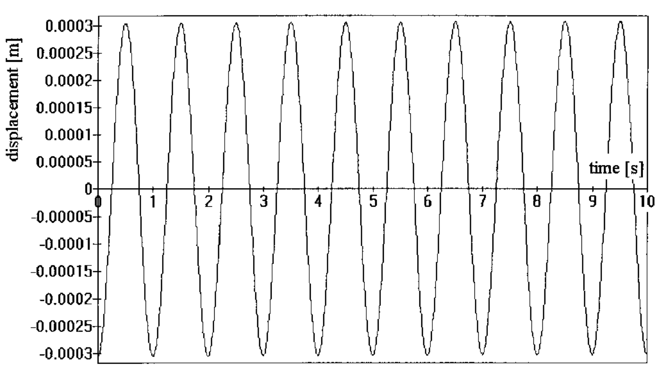

Figure 4. Time history of the system for evaluated initial conditions

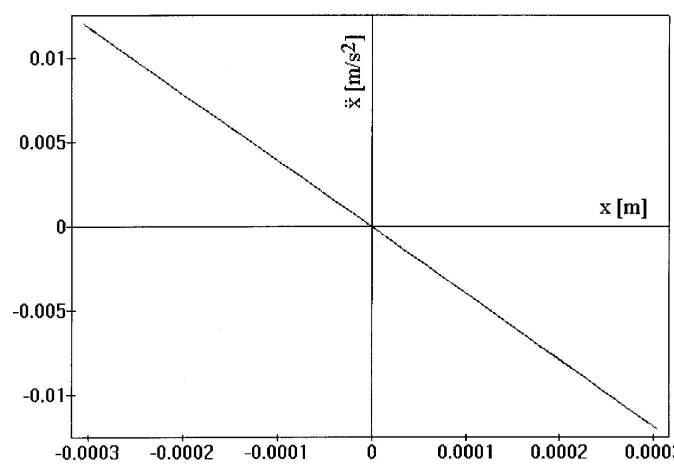

Figure 7. Plot of $x - \ddot{x}$ for the periodic response of the system