Orthogonal Eigenstructure Control for Vibration Suppression

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Orthogonal eigenstructure control is a novel active control method for vibration suppression in multi-input multi-output linear systems. This method is based on finding an output feedback control gain matrix in such a way that the closed-loop eigenvectors are almost orthogonal to the open-loop ones. Singular value decomposition is used to find the matrix, which spans the null space of the closed-loop eigenvectors. This matrix has a unique property that has been used in this new method. This unique property, which has been proved here, can be used to regenerate the open-loop system by finding a coefficient vector, which leads to a zero gain matrix. Also several vectors, which are orthogonal to the open-loop eigenvectors, can be found simultaneously. The proposed method does not need any trial and error procedure and eliminates not only the need to specify any location or area for the closed-loop eigenvalues but also the requirements of defining the desired eigenvectors. This method determines a set of limited number of closed-loop systems. Also, the elimination of the extra constraints on the locations of the closed-loop poles prevents the excessive force in actuators. [DOI: 10.1115/1.4000598]

Keywords: orthogonal eigenstructure control, eigenstructure assignment, active vibration cancellation, vibration control

1 Introduction

The idea of eigenstructure assignment (ESA) is given by Moore [1] and Clarke and Shelley [2]. He characterized the class of all eigenvector sets related to a distinct set of closed-loop eigenvalues using state feedback. Therefore, a control problem of eigenvalue placement for a multi-input multi-output (MIMO) system, which had been introduced earlier by Wonham [3], had been redefined to both placement of eigenvalues in desired locations and choosing a set of associated eigenvectors from a class of possible eigenvectors. This gives a considerable freedom to the control design, which Moore had presented as a design scheme. He explained that the speed of response is determined by the assigned closed-loop eigenvalues, while the shape of the response is related to the assigned eigenvectors. Klein and Moore [4] also presented an algorithm for nondistinct closed-loop eigenvalues and their related eigenvectors.

Srinathkumar [5] used a parametric output feedback scheme and derived sufficient conditions to assign a set of arbitrary distinct eigenvalues. A parametric approach introduced by Fahmy using the differentiation of the determinant of the combined system matrices. This method identifies the class of achievable eigenvectors and describes explicitly the generalized eigenvectors associated with the assigned eigenvalues. It also includes the entire closed-loop eigenstructure of the linear MIMO system [6,7]. Andry et al. [8] studied constrained output feedback to reduce the controller complexity by setting some of the elements of the feedback gain matrix to zero. The eigenstructure assignment by determining a minimum distance, considering the fact that the desired eigenvectors will not usually lie in the corresponding achievable subspaces had been studied by Calvo-Ramon [9].

It was Cunningham who first used singular value decomposition (SVD) to find the null space for the achievable eigenvector subspace. In his output feedback control method, the basis vectors were optimally combined to minimize the error between achievable and desirable eigenvectors. This method was the first practical method of eigenstructure assignment in order to have a desirable transient response behavior [10]. Using SVD, a finite number of actuators is needed to shape the eigenvectors of the system [11].

A different method of vibration confinement, based on applying a distributed feedback for one- and two-dimensional structures, such as uniform strings and beams, has been proposed by Choura and co-worker [12,13]. In this method, feedback control alters the mode shapes to exponentially decaying functions of space. In fact, feedback makes the settling time of some parts of the system to be faster while making the rate of settling of the other parts slower. This method is basically an inverse eigenvalue problem that the designer finds the best geometry and material properties to confine the energy. As a result, the mass, damping, and stiffness matrices for the closed-loop system are highly populated and therefore the number of actuators and sensors is equal to the number of states. To address the problem of limited actuators, a new method by partitioning the system in two subsystems has been introduced by Choura and Yigit in Ref. [11]. All the sensitive parts and the neighboring areas are to be controlled by altering the eigenvectors. The rest of the insensitive parts of the system are stabilized by different actuators. Even though both parts are being controlled; however, more attention is paid to the sensitive areas that are located in the first subsystem such that settling time for this subsystem is faster. This way, the original vibratory modes are being converted into the modes that let the vibration energy remain in the insensitive parts. A case study of this subject on the axial vibration confinement of the rods is presented in Ref. [14]. A similar method for linear time varying systems has been proposed in Ref. [15].

Shelly et al. studied the absolute displacement in first and second order systems, because the existing eigenstructure assignment would not have a control on them. They showed that it is not possible to tell if the absolute displacements in a system are increased, decreased, or remained intact just by changing the system's eigenvectors [16]. Furthermore, they introduced a mode localization technique called eigenvector scaling while studying the time domain response of the system. This method changes specific elements of each eigenvector in order to uniformly decrease the relative displacement of the corresponding areas in the system [17]. They showed analytically that absolute displacements in iso-

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Contributed by the Technical Committee on Vibration and Sound of ASME for publication in the Journal of Vibration and Acoustics. Manuscript received March 6, 2007; final manuscript received September 12, 2008; published online January 8, 2010. Editor: Kon-Well Wang.

lated areas can be reduced by eigenvector shaping, regardless of the type of the disturbance. Some experimental results have been reported in Refs. [2,18,19]. The eigenstructure shaping method is an active control method and is basically regenerating the behavior of the system when passive mode localization happens by scaling and reforming part or all of the system mode shapes. Since all the shape modes are scaled in the same way, vibration confinement of the system is not affected by the type of disturbance. An application of this method is also reported in Ref. [20]. One of the drawbacks of the uniform scaling is that the number of needed actuator/sensor that has to be equivalent to the number of coupled modes of the system. It means that the action between neighboring systems has the key role in the number of actuators/sensors that are needed. SVD-eigenvector shaping has been introduced and used as a solution to the problem of limited actuators/sensors [21]. This method uses a Moore–Penrose generalized left inverse and produces the closest eigenvector in least square sense to the desired ones, since it gives the minimum Euclidean two-norm error. This method allows use of fewer pairs of actuators and sensors than previous methods [21].

An active-passive hybrid vibration confinement system using piezoelectric network actuators has been proposed by Tang and Wang [22,23]. Instead of the mechanical parts, the passive elements of the systems are the circuit inductors and resistors. This method finds optimal eigenvectors using the Rayleigh principle by minimizing the ratio of modal energy at the concerned area to the modal energy of the whole structure using an auxiliary eigenvalue problem. Therefore the need for preselecting the closed-loop eigenvectors is eliminated, and the problem of closeness of the desired and achievable eigenvectors does not exist. A case study of this method has been presented in Ref. [24].

Predetermination of the desired eigenvector components can cause unsatisfactory performance if a match between components of the desired and achievable eigenvectors happens in the unimportant degrees of freedom [22,23]. Considering the problem of movement of neighborhood of the closed-loop eigenvalues, an eigenstructure method for constrained state or output feedback has been presented by Slater and Zhang [25]. They showed when the eigenvectors are the only parameters that are required to be changed, the control efforts are not necessarily minimized if the closed-loop eigenvalues are being forced to be close to the open-loop eigenvalues. In fact, a large change in eigenvectors may need a large movement of the eigenvalues to minimize the feedback gains. They also showed that closed-loop eigenvalues, and eigenvectors have to be consistent in order to avoid the large control efforts. Also, they proposed that since there is no method to have closed-loop eigenvectors and eigenvalues consistent, the minimum number of constraints should be imposed to the eigenvectors' elements in order to have a reasonable control effort.

The proposed method in this paper does not require specifying the locations for the closed-loop eigenvalues. The closed-loop system has eigenvalues, which are different from open-loop eigenvalues, and are consistent with the closed-loop eigenvectors.

The orthogonal eigenstructure control [26] introduced in this paper uses output feedback for controlling vibrations in multi-input multi-output linear systems. This method finds the closed-loop eigenstructures within the achievable eigenvectors set, such that their eigenvectors are orthogonal to the open-loop eigenvectors. Most of the known eigenstructure assignment methods require a predetermination of the closed-loop eigenstructure or at least eigenvectors. A prior knowledge of the desired closed-loop system behavior in terms of the elements of its eigenvectors is not an easy task and from a practical point of view is very challenging. Predicting a desirable shape for the eigenvectors of a complicated system does not have a straightforward procedure. Especially, for the continuous system, increasing the model degrees of freedom makes the task of defining the desirable shape for eigenvectors even harder, since there is no one-to-one relation between the states and the elements of the eigenvectors. The proposed method does not need a predetermination of the closed-loop eigenstructure; so, a prior knowledge of the closed-loop system is not required. The flexibility of the available eigenstructure assignment methods which requires designers to specify the desired closed-loop eigenvectors leads to an error due to the difference between the desirable and achievable eigenvectors. This method finds the achievable eigenvectors for the closed-loop system, which are orthogonal to the open-loop eigenvectors. As a result, there is virtually no limitation on the number of pairs of actuators and sensors, as well as the degrees of freedom of the model itself. The eigenvectors of the closed-loop system are achievable eigenvectors and the closed-loop eigenvalues are consistent with them.

Orthogonal eigenstructure control is based on proving an interesting property in the null space generated by (SVD). The upper part of the matrix that spans the null space is known as the basis for the eigenvectors of the closed-loop system. This submatrix has the same row dimension as state matrix of the system does. Multiplying the conjugate transpose of this matrix to itself is basically the norms of the eigenvectors of the closed-loop system. This matrix product can be expressed as the modal energy of the system as well. This product is a Hermitian matrix, which has a unique property of having zero and one eigenvalues. Multiplying the conjugate transpose of the lower part to itself leads to a Hermitian matrix. This matrix has identical eigenvectors to the similar product of the upper part one. However, summation of the eigenvalue matrices associated with the lower and upper parts is the identity matrix of compatible dimension.

Using this property, the open-loop system can be regenerated by eigenvector associated with the unity eigenvalue of upper part product, which is identical to the one of zero eigenvalue of lower part product. If the other eigenvectors are chosen, closed-loop systems with eigenvectors orthogonal to the open-loop eigenvectors can be found.

2 Problem Statement and Finding the Null Space

To define the problem, a multi-input multi-output second order linear system of equations in matrix notation is considered

$$M\ddot{q} + C_d\dot{q} + K_s q = F_i u_i + F_d u_d \tag{1}$$

where M, C_d, and K_s are $n \times n$ mass, damping, and stiffness matrices of the system, respectively. The number of collocated actuators and sensors is $m \geq 2$. F_i and F_d are the respective control input and the disturbance input matrices, while q, $\dot{q}$, and $\ddot{q}$ are the displacement, velocity, and acceleration of the modes of the system. u_i and u_d are the external control and the external disturbance vectors. The equation of the motion of this system in the state space form can be written as

$$\dot{x} = Ax + Bu + Ef \tag{2}$$

where A is the $2n \times 2n$ state matrix, B is the $2n \times m$ input matrix, and E is the disturbance input matrix with $2n$ rows. Their definitions can be found in Appendix A. u is the input vector of dimension m, and f is the disturbance vector. x is the $2n \times 1$ state vector, and $\dot{x}$ is the time derivative of the state vector.

The state vector is

$$x = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} \tag{3}$$

Also, the output equation for the system can be written as

$$y = Cx \tag{4}$$

where C is the $m \times 2n$ output matrix, and y is the $m \times 1$ output vector. In order to have an output feedback control, the input control force can be written as

$$u = Ky \tag{5}$$

where K is the $m \times m$ feedback gain matrix. The closed-loop equation of the motion can be written as

$$\dot{x} = (A + BKC)x + Ef \quad (6)$$

The actuators and sensors are assumed to be collocated. The eigenstructure assignment finds the control gain matrix K in order to cancel the effect of the disturbance f in the areas that are needed to be isolated. For this reason, the eigenvalues and eigenvectors of the closed-loop system need to be determined simultaneously while the gain matrix $[K]$ is determined. Therefore, the following eigenvalue problem has to be solved:

$$(A + BKC)\phi_i = \lambda_i \phi_i, \quad i = 1, \ldots, 2n \quad (7)$$

for K, λ_i, and ϕ_i, where ϕ_i are the closed-loop eigenvectors of the system. λ_i are generally the closed-loop eigenvalues associated with ϕ_i; however, in this proposed method are the open-loop eigenvalues and we call them operating eigenvalues. The latter equation can be written in the matrix form as

$$[A - \lambda_i I | B] \begin{Bmatrix} \phi_i \\ KC\phi_i \end{Bmatrix} = 0, \quad i = 1, \ldots, 2n \quad (8)$$

where I is the $2n \times 2n$ identity matrix. It can be seen that the vector $\left\{ \begin{smallmatrix} \phi_i \\ KC\phi_i \end{smallmatrix} \right\}$ spans the null space of the matrix $S_{\lambda i} = [A - \lambda_i I | B]$.

Applying SVD to $S_{\lambda i}$, we can write

$$S_{\lambda i} = [A - \lambda_i I | B]_{2n \times (2n+m)} = [U_i]_{2n \times 2n} [\Sigma_i | 0_{2n \times m}]_{2n \times (2n+m)} [V_i^*]_{(2n+m) \times (2n+m)} \quad (9)$$

λ_i in Eq. (9) are the operating eigenvalues. The number of operating eigenvalues is the same as the number of the required pairs of actuators and sensors m. The first m open-loop eigenvalues with largest negative real parts are chosen to be the operating eigenvalues. This leads to have more desirable closed-loop systems. U_i and V_i are the left and right orthonormal matrices, respectively. V_i^* is the conjugate transpose of the complex matrix V_i. Index i is used to specify the equation for the ith closed-loop eigenvalue. Using the orthonormal property, we can write

$$U_i^* U_i = I_{2n \times 2n} \quad (10)$$

and

$$V_i^* V_i = I_{(2n+m) \times (2n+m)} \quad (11)$$

V_i can be partitioned as

$$[V_i]_{(2n+m) \times (2n+m)} = \begin{bmatrix} [V_{11}^i]_{2n \times 2n} & [V_{12}^i]_{2n \times m} \\ [V_{21}^i]_{m \times 2n} & [V_{22}^i]_{m \times m} \end{bmatrix} \quad (12)$$

It is shown in Appendix B that the second column block of the V_i spans the null space of the $S_{\lambda i}$.

Any linear combination of m columns of V_{12}^i is an achievable eigenvector of the closed-loop system. While the problem of eigenstructure assignment can be reduced to the problem of finding coefficient vector r^i, different methods have different approaches to find r^i.

If the desirable eigenvectors of the system due to a particular vector r^i is

$$\phi_i^a = V_{12}^i r^i \quad (13)$$

then the corresponding control gain matrix K can be found using the following equation:

$$KC\phi_i^a = V_{22}^i r^i \quad (14)$$

Available methods define a desired eigenvector ϕ_i^d for the system using different approaches such as eigenvector shaping. Reference [27] uses a Moore–Penrose inverse of V_{12}^i to find the required r^i. The limitation of those methods is the difference between the controlled eigenvectors and the desired eigenvectors because the Moore–Penrose inverse of a vector gives the closest possible vector in sense of the least square or Euclidean two-norm. Therefore, in general, there is always a distance between the desired and controlled eigenvectors [27]. Figure 1 illustrates the difference between achievable and desirable eigenvectors schematically.

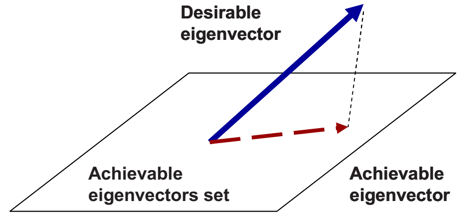

Fig. 1 The difference between achievable and desirable eigenvectors

As it stated earlier, the orthogonal eigenstructure control regenerates the open-loop system by finding the open-loop eigenvectors and the orthogonal vectors to them. That is, the open-loop eigenvectors of the operating eigenvalues are the intersections of the open-loop eigenvectors set and the sets of achievable eigenvectors. It is shown in Fig. 2.

3 Orthogonal Eigenstructure Control

The norm of the ith eigenvector of the closed-loop system can be written as

$$E_i = r^{i*} V_{12}^{i*} V_{12}^i r^i \quad (15)$$

since it can be seen from Eq. (13) that an achievable eigenvector can be written as $\phi_i^a = V_{12}^i r^i$. The null space of the closed-loop eigenvectors associated with the operating eigenvalue λ_i is spanned by N^i

$$N^i = \begin{Bmatrix} V_{12}^i \\ V_{22}^i \end{Bmatrix} \quad (16)$$

Norm of N^i is equal to one since N^i contains the basis of the null space

$$\|N^i\|_2 = 1 \quad (17)$$

Therefore, any row block of N^i has a norm of equal or less than unity. Hence, the magnitudes of their singular values belong to the interval [0 1] [28]

$$\|V_{12}^i\| \leq 1 \quad (18)$$

We apply SVD to V_{12}^i

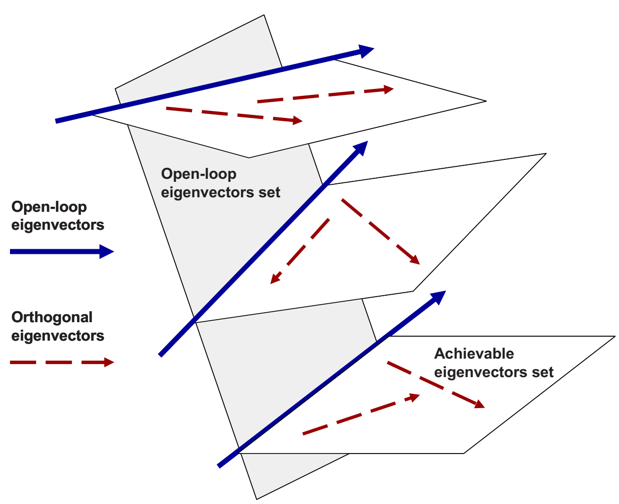

Fig. 2 Open-loop eigenvectors are the intersections of the open-loop and achievable eigenvectors sets

$$V_{12}^{i} = \bar{V}^{i}\bar{S}^{i}\bar{U}^{i*} \tag{19}$$

while singular values of $\bar{S}^{i}$ lie in the interval [0 1]. Also, it can be written

$$V_{12}^{i*} = \bar{U}^{i}\bar{S}^{i*}\bar{V}^{i*} \tag{20}$$

Multiplying both sides of Eq. (20) by Eq. (19) yields

$$V_{12}^{i*}V_{12}^{i} = \bar{U}^{i}\bar{S}^{i*}\bar{V}^{i*}\bar{V}^{i}\bar{S}^{i}\bar{U}^{i*} \tag{21}$$

Since $\bar{V}^{i}$ is an orthonormal matrix, it can be seen that

$$\bar{V}^{i*}\bar{V}^{i} = I \tag{22}$$

Therefore,

$$V_{12}^{i*}V_{12}^{i} = \bar{U}^{i}\bar{S}^{i*}\bar{S}^{i}\bar{U}^{i*} = \bar{U}^{i}|\bar{S}^{i}|^{2}\bar{U}^{i*} = \bar{U}^{i}\Lambda^{i}\bar{U}^{i*} \tag{23}$$

$\bar{\Lambda}_{i}$ is the matrix of eigenvalues and $\bar{U}^{i}$ is the matrix of eigenvectors of $V_{12}^{i*}V_{12}^{i}$. Equation (23) clearly shows that the eigenvalues of the Hermitian product $V_{12}^{i*}V_{12}^{i}$ belong to the interval [0 1], since the absolute values of the singular values of $\bar{S}^{i}$ belong to this interval.

Interestingly, $V_{22}^{i*}V_{22}^{i}$ has the same eigenvectors as $V_{12}^{i*}V_{12}^{i}$, but its eigenvalues are different. More precisely, the summation of the eigenvalues of $V_{12}^{i*}V_{12}^{i}$ and $V_{22}^{i*}V_{22}^{i}$ associated with similar eigenvectors are unity.

THEOREM. *Consider a* $2n\times m$ nonsquare matrix $S_{\lambda i}$ and $2n \geq m$. The null space of this matrix is spanned by the columns of a $(2n+m)\times m$ matrix N^{i}. V_{12}^{i} is the upper $2n\times m$ submatrix of N^{i}, and V_{22}^{i} is the lower $m\times m$ submatrix of N^{i}. $V_{12}^{i*}V_{12}^{i}$ and $V_{22}^{i*}V_{22}^{i}$ have identical $m\times m$ eigenvector matrices and the summation of their ordered eigenvalue matrices is an $m\times m$ identity matrix.

Proof. From Eq. (17) we can write

$$N_{i}^{*}N_{i} = I \tag{24}$$

which can be expanded as

$$[V_{12}^{i}]_{2n\times 2n}^{*}[V_{12}^{i}]_{2n\times 2n} + [V_{22}^{i}]_{m\times m}^{*}[V_{22}^{i}]_{m\times m} = I \tag{25}$$

Eq. (25) can be rewritten using eigenvalue decomposition

$$\bar{U}^{i}\bar{\Lambda}^{i}\bar{U}^{i*} + \bar{U}_{w}^{i}\bar{\Lambda}_{w}^{i}\bar{U}_{w}^{i*} = I \tag{26}$$

where $\bar{\Lambda}_{w}^{i}$ and $\bar{U}_{w}^{i}$ are the eigenvalue and eigenvector matrices of $V_{22}^{i*}V_{22}^{i}$. Premultiplying Eq. (26) by $\bar{U}^{i*}$ and postmultiplying it by $\bar{U}^{i}$ yields

$$\bar{\Lambda}^{i} + \bar{U}^{i*}\bar{U}_{w}^{i}(\bar{\Lambda}_{w}^{i})\bar{U}_{w}^{i*}\bar{U}^{i} = \bar{U}^{i*}I\bar{U}^{i} = I \tag{27}$$

Re-arranging Eq. (27)

$$(\bar{U}^{i*}\bar{U}_{w}^{i})(\bar{\Lambda}_{w}^{i})(\bar{U}^{i*}\bar{U}_{w}^{i})^{*} = I - \bar{\Lambda}^{i} \tag{28}$$

The left hand side of Eq. (28) is basically an eigenvalue decomposition of the diagonal matrix $I-\bar{\Lambda}^{i}$. Since the eigenvalue matrix of a diagonal matrix is the matrix itself, we may write

$$\bar{\Lambda}_{w}^{i} = I - \bar{\Lambda}^{i} \tag{29}$$

or

$$\bar{\Lambda}_{w}^{i} + \bar{\Lambda}^{i} = I \tag{30}$$

Equation (28) holds if

$$\bar{U}^{i*}\bar{U}_{w}^{i} = I \tag{31}$$

which concludes

$$\bar{U}^{i} = \bar{U}_{w}^{i} \tag{32}$$

□

If the eigenvector $\bar{U}_{J}^{i}$ associated with a unity eigenvalue of $V_{12}^{i*}V_{12}^{i}$ is substituted as r^{i} in Eq. (15), then $E^{i}=1$. Eigenvalue decomposition of $V_{12}^{i*}V_{12}^{i}$ implies

$$V_{12}^{i*}V_{12}^{i} = \bar{U}^{i}\bar{\Lambda}^{i}\bar{U}^{i*} \tag{33}$$

The latter equation can be re-arranged as

$$\bar{U}^{i*}(V_{12}^{i*}V_{12}^{i})\bar{U}^{i} = \bar{\Lambda}^{i} \tag{34}$$

Choosing the eigenvalue equal to unity and its corresponding eigenvector $\bar{U}_{J}^{i}$

$$\bar{U}_{J}^{i*}(V_{12}^{i*}V_{12}^{i})\bar{U}_{J}^{i} = 1 \tag{35}$$

This equation shows that $V_{12}^{i}\bar{U}_{J}^{i}$ is identical to the eigenvector corresponding to the operating eigenvalue of the open-loop, since its norm is one and the gain associated with this eigenvector is zero.

Also, Eq. (14) shows that corresponding vector of the gain matrix becomes zero. $\bar{U}_{J}^{i}$ is the eigenvector corresponding to eigenvalue zero of $V_{22}^{i*}V_{22}^{i}$ (unity eigenvalue for $V_{12}^{i*}V_{12}^{i}$). Using the eigenvalue problem equation for $V_{22}^{i*}V_{22}^{i}$ and substituting $\bar{U}_{J}^{i}$, we may write

$$\bar{U}_{J}^{i*}(V_{22}^{i*}V_{22}^{i})\bar{U}_{J}^{i} = 0 \tag{36}$$

Rewriting Eq. (36)

$$(V_{22}^{i}\bar{U}_{J}^{i})^{*}(V_{22}^{i}\bar{U}_{J}^{i}) = 0 \tag{37}$$

Eq. (37) holds if $V_{22}^{i}\bar{U}_{J}^{i}=0$

$$KC\phi_{i}^{a} = V_{22}^{i}r^{i} = V_{22}^{i}\bar{U}_{J}^{i} = 0 \tag{38}$$

That results in a zero gain matrix K. Since there is no control gain, the open-loop system has been regenerated.

Using the fact that other eigenvectors associated with nonunity eigenvalues of $V_{12}^{i*}V_{12}^{i}$ are orthogonal to the eigenvectors of the unity eigenvalue of $V_{12}^{i*}V_{12}^{i}$, closed-loop eigenvectors can be found knowing they are orthogonal to the open-loop ones.

The jth nonunity eigenvalue of $V_{12}^{i*}V_{12}^{i}$ is several order of magnitudes smaller than unity and their eigenvectors are $\bar{U}_{j}^{i}$. $\bar{U}_{j}^{i}$ is jth column of eigenvector matrix $\bar{U}^{i}$. If $\bar{U}_{j}^{i}$ is substituted in r^{i}, the following equation can be concluded:

$$r^{i*}V_{12}^{i*}V_{12}^{i}r^{i} = \bar{\lambda}_{j}^{i} \leq 1 \tag{39}$$

Comparing Eq. (39) to closed-loop eigenvector norm in Eq. (24), it can be seen that for the coefficient vector $r^{i}=\bar{U}_{j}^{i}$, the norm of the modes corresponding to the ith mode are lowered in comparison to the open-loop ones

$$E = \bar{U}_{j}^{i*}V_{12}^{i*}V_{12}^{i}\bar{U}_{j}^{i} \leq 1 \tag{40}$$

Appending all the calculated closed-loop eigenvectors for all the modes that have been calculated, one can write the following matrices:

$$V = [V_{12}^{1}r^{1} \cdots V_{12}^{m}r^{m}] \tag{41}$$

$$W = [V_{22}^{1}r^{1} \cdots V_{22}^{m}r^{m}] \tag{42}$$

Feedback gain matrix K can be found from Eq. (14)

$$K = W(CV)^{-1} \tag{43}$$

The state matrix for the closed-loop system is defined as

$$A_{c} = A + BKC \tag{44}$$

It has to be noted that it is possible to choose the eigenvectors associated with unity eigenvalues of $V_{12}^{i*}V_{12}^{i}$ as the coefficient vectors r^{i}. It means that the algorithm is not replacing those eigenvectors with an orthogonal one. It leads to the zero columns in W

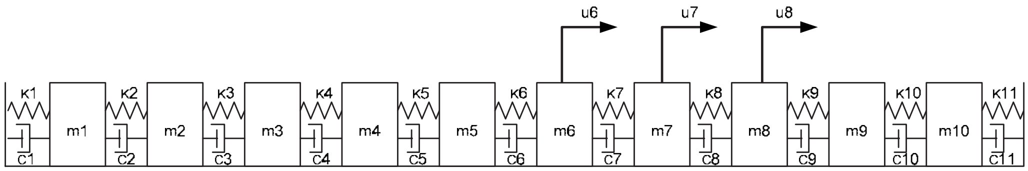

Fig. 3 The system of ten masses with interconnecting springs and dampers

and K will not have a full rank. As a result the actuation forces are linearly dependent.

The trace of a square matrix is the summation of its eigenvalues. Since the matrix product BKC has zero elements on its diagonal, therefore, the trace of BKC is zero. As a result the summations as well as the average of the eigenvalues for the open-loop and closed-loop systems are equal.

4 Performance Criteria In Orthogonal Eigenstructure Control

Let's rewriting Eq. (34) in an expanded form

$$\bar{U}^{i*}V_{12}^{i*}V_{12}^{i}\bar{U}^{i} = \begin{bmatrix} \bar{\lambda}_1^i & & & & \\ & \bar{\lambda}_2^i & & 0 & \\ & & \ddots & & \\ & 0 & & \bar{\lambda}_{m-1}^i & \\ & & & & \bar{\lambda}_m^i \end{bmatrix} \quad (45)$$

Among all the eigenvalues $\bar{\lambda}_j^i$ $j=1,2,\ldots,m$, there is one eigenvalue equal to unity and $m-1$ that are zero. If the eigenvector associated with all the unity eigenvalues are selected as the coefficient vectors, the open-loop system is just regenerated. There are m options for choosing the eigenvalues and corresponding eigenvectors of $V_{12}^{i*}V_{12}^{i}$ to be substituted in r^i. Obviously, when there are m inputs to the system, the number of eigenvectors that have to be found is m, while there are m choices for coefficients vectors. Excluding the regenerated open-loop system, there are m^m-1 options for orthogonal eigenstructure control. Figure 2 shows the open-loop eigenvectors and the achievable eigenvectors of a closed-loop system with three collocated actuators and sensors. For each open-loop eigenvector, two orthogonal eigenvectors can be found that lie within achievable eigenvectors set. If some of the eigenvectors have not been changed, a closed-loop system still can be found. As a result, there are $3^3-1=26$ closed-loop systems as the outcomes of this control method.

For a controlled system with m actuators, all the possible answers for one set of the closed-loop eigenstructure can be written using Eq. (40)

$$\bar{U}_j^{i*}V_{12}^{i*}V_{12}^{i}\bar{U}_j^{i} = \bar{\lambda}_j^i, \quad j=1,2,\ldots,m \quad (46)$$

Equation (46) holds for all the eigenvalues and their corresponding eigenvectors. It is seen that there are m^m-1 possible configurations for the closed-loop systems. The orthogonal eigenstructure control has the ability to identify these configurations. The most desirable solution is the one that has the shortest settling time and smallest overshoot of the states within the isolated region. To apply these criteria, the closed-loop system with the smallest phase plane of the states at the isolated region can be chosen as the closed-loop system.

5 Procedure of Orthogonal Eigenstructure Control

This procedure defines the proposed output feedback control of orthogonal eigenstructure control as follows.

1. Define state space realization of the system similar to Eq. (4).
2. Determine the m eigenvalues of the open-loop system with negative real parts having greatest absolute values, where m is the number of the pairs of actuators and sensors.
3. Using the ith λ_i, define nonsquare matrix $S_{\lambda i}$ as Eq. (9).
4. Find singular value decomposition of $S_{\lambda i}$ generated from step 3 using Eq. (9).
5. Partition the right unitary matrix V_i and define V_{12}^i and V_{22}^i, as described in Eq. (12).
6. Calculate $V_{12}^{i*}V_{12}^{i}$.
7. Find the eigenvalue matrix $\bar{\Lambda}^i$ and eigenvector matrix $\bar{U}^i$ for Hermitian matrix $V_{12}^{i*}V_{12}^{i}$, using Eq. (23).
8. Pick the jth eigenvector $\bar{U}_j^i$ associated with the jth eigenvalue λ_j^i. Define $r^i=\bar{U}_j^i$ similar to Eq. (38).
9. Calculate $v^i=V_{12}^i r^i$ and $w^i=V_{22}^i r^i$.
10. Repeat Steps 3–9, m times and find v_i and w_i for all the operating eigenvalues.
11. Define matrices V and W by appending v^i and w^i resulting from steps 9 and 10, as defined in Eqs. (41) and (42).
12. Find the gain matrix K using Eq. (43).
13. Go to step 8, change the index j. Repeat the algorithm m^m times. Exclude the case that regenerates the open-loop system.
14. Choose the gain matrix K from m^m-1 closed-loop systems that produce the smallest phase plane for the isolated states of the system.

6 Case Study: A System With Three Collocated Actuators and Sensors

In this section, a simple lumped system with longitudinal vibration is considered and the orthogonal eigenstructure control has been applied in order to isolate the left side of the system from the vibration. The system consists of ten masses that are interconnected by springs and dampers, as indicated in Fig. 3.

It is assumed that all the masses are equal to 50 kg. Also, all the spring coefficients are identical and are equal to 1000 N/m. Damping coefficients are assumed to be 10 N s/m.

This system is considered to have three pairs of collocated actuators and sensors located on m_6, m_7, and m_8. All the element of B and C matrices are zero except

$$B(16,1) = B(17,2) = B(18,3) = -1/50$$

$$C(1,6) = C(2,7) = C(3,8) = 1$$

Operating eigenvalues, based on the step 2 of the procedure are $\lambda_1=-0.4919+8.8396i$, $\lambda_3=-0.4683+8.5692i$, and $\lambda_5=-0.4310+8.1246i$. Therefore the problem is to find the appropriate r^1, r^2, and r^3 for $V_{12}^1 r^1$, $V_{12}^2 r^2$, and $V_{12}^3 r^3$, respectively. Two different cases with different values for r^i associated with each operating eigenvalues are considered.

6.1 Case 1. $r^1=\bar{U}_3^1$, $r^2=\bar{U}_3^2$, and $r^3=\bar{U}_3^3$, which gives the zero gain matrix. The closed-loop system is a regeneration of the open-loop system.

6.2 Case 2. $r^1=\bar{U}_2^1$, $r^2=\bar{U}_1^2$, and $r^3=\bar{U}_1^3$, which generates the most desirable vibration isolation.

For case 1, with the first operating eigenvalue, λ_1, the following equations can be written

$$V_{12}^{1*}V_{12}^{1}=\begin{bmatrix}0.4137 & -0.3793 & 0.3140\\ -0.3793 & 0.3479 & -0.2881\\ 0.3140 & -0.2881 & 0.2387\end{bmatrix}$$

$$V_{12}^{1*}V_{12}^{1}=\bar{U}^{1}\bar{\Lambda}^{1}\bar{U}^{1*}=\begin{bmatrix}0.3723 & -0.6692 & 0.6431\\ 0.7982 & -0.1228 & -0.5898\\ 0.4737 & 0.7329 & 0.4884\end{bmatrix}\times\begin{bmatrix}0 & \cdots & 0\\ \vdots & 0.0002 & \vdots\\ 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}0.3723 & 0.7982 & 0.4737\\ -0.6692 & -0.1228 & 0.7329\\ 0.6431 & -0.5898 & 0.4884\end{bmatrix}$$

Also

$$V_{22}^{1*}V_{22}^{1}=\begin{bmatrix}0.5863 & 0.3793 & -0.3140\\ 0.3793 & 0.6521 & 0.2881\\ -0.3140 & 0.2881 & 0.7613\end{bmatrix}$$

$$V_{22}^{1*}V_{22}^{1}=\bar{U}^{1}\bar{\Lambda}_{w}^{1}\bar{U}^{1*}=\begin{bmatrix}0.3723 & -0.6692 & 0.6431\\ 0.7982 & -0.1228 & -0.5898\\ 0.4737 & 0.7329 & 0.4884\end{bmatrix}\times\begin{bmatrix}1 & \cdots & 0\\ \vdots & 0.9998 & \vdots\\ 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}0.3723 & 0.7982 & 0.4737\\ -0.6692 & -0.1228 & 0.7329\\ 0.6431 & -0.5898 & 0.4884\end{bmatrix}$$

It is seen that

$$V_{12}^{1*}V_{12}^{1}+V_{22}^{1*}V_{22}^{1}=\begin{bmatrix}0.4137 & -0.3793 & 0.3140\\ -0.3793 & 0.3479 & -0.2881\\ 0.3140 & -0.2881 & 0.2387\end{bmatrix}+\begin{bmatrix}0.5863 & 0.3793 & -0.3140\\ 0.3793 & 0.6521 & 0.2881\\ -0.3140 & 0.2881 & 0.7613\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

In addition, $V_{12}^{1*}V_{12}^{1}$ and $V_{22}^{1*}V_{22}^{1}$ have the same eigenvectors and their eigenvalues satisfy Eq. (30)

$$\bar{\Lambda}_{w}^{1}+\bar{\Lambda}^{1}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & 0.9998 & \vdots\\ 0 & \cdots & 0\end{bmatrix}+\begin{bmatrix}0 & \cdots & 0\\ \vdots & 0.0002 & \vdots\\ 0 & \cdots & 1\end{bmatrix}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & 1 & \vdots\\ 0 & \cdots & 1\end{bmatrix}$$

The first column of W becomes zero

$$V_{22}^{1}r^{1}=V_{22}^{1}\bar{U}_{3}^{1}=\begin{bmatrix}0.5872 & 0.3793 & -0.3151\\ 0.3785 & 0.6521 & 0.2890\\ -0.3135 & 0.2881 & 0.7606\end{bmatrix}\begin{bmatrix}0.6431\\ -0.5898\\ 0.4884\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

For the second operating eigenvalue, λ_3, the following equations can be written:

$$V_{12}^{3*}V_{12}^{3}=\begin{bmatrix}0.0445 & -0.1245 & 0.1636\\ -0.1245 & 0.3501 & -0.4604\\ 0.1636 & -0.4604 & 0.6058\end{bmatrix}$$

$$V_{12}^{3*}V_{12}^{3}=\bar{U}^{3}\bar{\Lambda}^{3}\bar{U}^{3*}=\begin{bmatrix}-0.4115 & -0.8868 & 0.2103\\ -0.7757 & 0.2196 & -0.5917\\ -0.4785 & 0.4066 & 0.7783\end{bmatrix}\times\begin{bmatrix}0 & \cdots & 0\\ \vdots & 0.0003 & \vdots\\ 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}-0.4115 & -0.7757 & -0.4785\\ -0.8868 & 0.2196 & 0.4066\\ 0.2103 & -0.5917 & 0.7783\end{bmatrix}$$

Also

$$V_{22}^{3*}V_{22}^{3}=\begin{bmatrix}0.9555 & 0.1245 & -0.1636\\ 0.1245 & 0.6499 & 0.4604\\ -0.1636 & 0.4604 & 0.3942\end{bmatrix}$$

$$V_{22}^{3*}V_{22}^{3}=\bar{U}^{3}\bar{\Lambda}_{w}^{3}\bar{U}^{3*}=\begin{bmatrix}-0.4115 & -0.8868 & 0.2103\\ -0.7757 & 0.2196 & -0.5917\\ -0.4785 & 0.4066 & 0.7783\end{bmatrix}\times\begin{bmatrix}1 & \cdots & 0\\ \vdots & 0.9997 & \vdots\\ 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}-0.4115 & -0.7757 & -0.4785\\ -0.8868 & 0.2196 & 0.4066\\ 0.2103 & -0.5917 & 0.7783\end{bmatrix}$$

It is seen that

$$V_{12}^{3*}V_{12}^{3}+V_{22}^{3*}V_{22}^{3}=\begin{bmatrix}0.0445 & -0.1245 & 0.1636\\ -0.1245 & 0.3501 & -0.4604\\ 0.1636 & -0.4604 & 0.6058\end{bmatrix}+\begin{bmatrix}0.9555 & 0.1245 & -0.1636\\ 0.1245 & 0.6499 & 0.4604\\ -0.1636 & 0.4604 & 0.3942\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

It can be seen that $V_{12}^{3*}V_{12}^{3}$ and $V_{22}^{3*}V_{22}^{3}$ have the same eigenvectors and their eigenvalues satisfy Eq. (30)

$$\bar{\Lambda}_{w}^{3}+\bar{\Lambda}^{3}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & 0.9997 & \vdots\\ 0 & \cdots & 0\end{bmatrix}+\begin{bmatrix}0 & \cdots & 0\\ \vdots & 0.0003 & \vdots\\ 0 & \cdots & 1\end{bmatrix}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & 1 & \vdots\\ 0 & \cdots & 1\end{bmatrix}$$

and the second column of W becomes zero

$$V_{22}^{3}r^{3}=V_{22}^{3}\bar{U}_{3}^{3}=\begin{bmatrix}0.9555 & 0.1245 & -0.1636\\ 0.1245 & 0.6499 & 0.4604\\ -0.1636 & 0.4604 & 0.3942\end{bmatrix}\begin{bmatrix}0.2103\\ -0.5917\\ 0.7783\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Similarly, for the third operating eigenvalue, λ_5, the following equations can be written:

$$V_{12}^{5*}V_{12}^{5}=\begin{bmatrix}0.6954 & -0.2180 & -0.4053\\ -0.2180 & 0.0684 & 0.1269\\ -0.4053 & 0.1269 & 0.2364\end{bmatrix}$$

$$V_{12}^{5*}V_{12}^{5}=\bar{U}^{5}\bar{\Lambda}^{5}\bar{U}^{5*}=\begin{bmatrix}-0.8339 & -0.2216 & 0.5054\\ 0.2614 & 0.6480 & 0.7154\\ 0.4861 & -0.7287 & 0.4825\end{bmatrix}\times\begin{bmatrix}0 & \cdots & 0\\ \vdots & 0.0003 & \vdots\\ 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}-0.8339 & 0.2614 & 0.4861\\ -0.2216 & 0.6480 & -0.7287\\ 0.5054 & 0.7154 & 0.4825\end{bmatrix}$$

Also

$$V_{22}^{5*}V_{22}^{5}=\begin{bmatrix}0.3046 & 0.2180 & 0.4053\\ 0.2180 & 0.9316 & -0.1269\\ 0.4053 & -0.1269 & 0.7636\end{bmatrix}$$

$$V_{22}^{5*}V_{22}^{5}=\bar{U}^{5}\bar{\Lambda}_{w}^{5}\bar{U}^{5*}=\begin{bmatrix}-0.8339 & -0.2216 & 0.5054\\ 0.2614 & 0.6480 & 0.7154\\ 0.4861 & -0.7287 & 0.4825\end{bmatrix}\times\begin{bmatrix}1 & \cdots & 0\\ \vdots & 0.9997 & \vdots\\ 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}-0.8339 & 0.2614 & 0.4861\\ -0.2216 & 0.6480 & -0.7287\\ 0.5054 & 0.7154 & 0.4825\end{bmatrix}$$

and

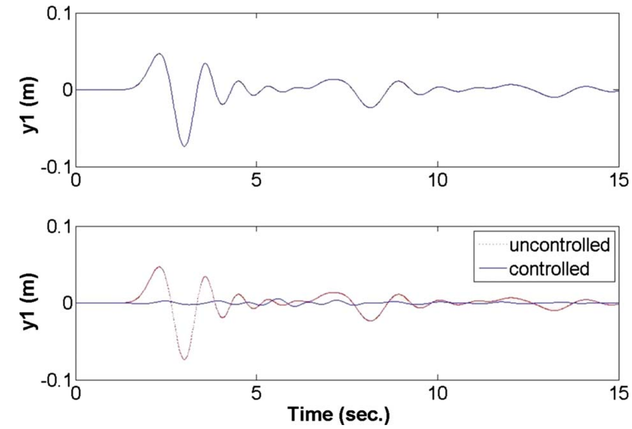

Fig. 4 Displacement of m_1 with operating eigenvalues (λ_1, λ_3, and λ_5) due to an impulse input on m_{10}; Case 1: $r^1=\bar{U}_3^1$, $r^2=\bar{U}_3^2$, and $r^3=\bar{U}_3^3$; Case 2: $r^1=\bar{U}_2^1$, $r^2=\bar{U}_1^2$, and $r^3=\bar{U}_1^3$

$$V_{12}^{5*}V_{12}^5 + V_{22}^{5*}V_{22}^5 = \begin{bmatrix} 0.6954 & -0.2180 & -0.4053 \\ -0.2180 & 0.0684 & 0.1269 \\ -0.4053 & 0.1269 & 0.2364 \end{bmatrix} + \begin{bmatrix} 0.3046 & 0.2180 & 0.4053 \\ 0.2180 & 0.9316 & -0.1269 \\ 0.4053 & -0.1269 & 0.7636 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similar to the previous ones, it is shown that $V_{12}^{5*}V_{12}^5$ and $V_{22}^{5*}V_{22}^5$ have the same eigenvectors and their eigenvalues satisfy Eq. (30)

$$\bar{\Lambda}_w^5 + \bar{\Lambda}^5 = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 0.9997 & \vdots \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 0.0003 & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Also, the third column of W becomes zero

$$V_{22}^5 r^5 = V_{22}^5 \bar{U}_3^5 = \begin{bmatrix} 0.3046 & 0.2180 & 0.4053 \\ 0.2180 & 0.9316 & -0.1269 \\ 0.4053 & -0.1269 & 0.7636 \end{bmatrix} \begin{bmatrix} 0.5054 \\ 0.7154 \\ 0.4825 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Similar results have been found for $V_{22}^2 r^2$ and $V_{22}^3 r^3$, if r^2 and r^3 are the eigenvectors associated with unity eigenvalue of $V_{12}^{2*}V_{12}^2$ and $V_{12}^{3*}V_{12}^3$ (eigenvalue 0 of $V_{22}^{2*}V_{22}^2$ and $V_{22}^{3*}V_{22}^3$), respectively. As a result, according to Eqs. (42) and (43), the gain matrix becomes zero. Therefore, the open-loop and closed-loop systems are identical, as it can be seen in Fig. 4.

For case 2, which is the most desirable closed-loop system, the following equations can be written:

$$V_{22}^1 r^1 = V_{22}^1 \bar{U}_2^1 = \begin{bmatrix} 0.5872 & 0.3793 & -0.3151 \\ 0.3785 & 0.6521 & 0.2890 \\ -0.3135 & 0.2881 & 0.7606 \end{bmatrix} \begin{bmatrix} -0.6692 \\ -0.1228 \\ 0.7329 \end{bmatrix} = \begin{bmatrix} -0.6705 \\ -0.1216 \\ 0.7318 \end{bmatrix}$$

$$V_{22}^3 r^3 = V_{22}^3 \bar{U}_1^3 = \begin{bmatrix} 0.9555 & 0.1245 & -0.1636 \\ 0.1245 & 0.6499 & 0.4604 \\ -0.1636 & 0.4604 & 0.3942 \end{bmatrix} \begin{bmatrix} -0.4115 \\ -0.7757 \\ -0.4785 \end{bmatrix} = \begin{bmatrix} -0.4109 \\ -0.7772 \\ -0.4765 \end{bmatrix}$$

and

$$V_{22}^5 r^5 = V_{22}^5 \bar{U}_1^5 = \begin{bmatrix} 0.3046 & 0.2180 & 0.4053 \\ 0.2180 & 0.9316 & -0.1269 \\ 0.4053 & -0.1269 & 0.7636 \end{bmatrix} \begin{bmatrix} -0.8339 \\ 0.2614 \\ 0.4861 \end{bmatrix} = \begin{bmatrix} -0.5074 \\ -0.7147 \\ -0.4813 \end{bmatrix}$$

W in Eq. (42) is found by combining the previous results as

$$W = \begin{bmatrix} -0.6705 & -0.4109 & -0.5074 \\ -0.1216 & -0.7772 & -0.7147 \\ 0.7318 & -0.4765 & -0.4813 \end{bmatrix}$$

and the real gain matrix is

$$K = W(CV)^{-1} = 1.0 \times 10^3 \begin{bmatrix} -1.3205 & -0.8670 & -0.1337 \\ 1.5734 & -1.5268 & 0.1758 \\ 0.7264 & -1.0190 & -0.8965 \end{bmatrix}$$

A unit impulse input is applied to m_{10} and the time histories of the displacement of m_1 are presented in Fig. 4 for different cases. It is observed that the closed-loop response of y_1 is the same as the open-loop response in case 1, while vibration is isolated in case 2.

Case 2 is the output of this procedure, which has satisfactory characteristics in terms of short settling time and small overshoot with respect to the open-loop system. To find which one of the outputs of the algorithm is the most desirable solution, the system with the smallest phase plane can be selected. The phase planes of one of the masses in the isolated area, i.e., m_1, for eight closed-loop systems are presented in Fig. 5. The smallest phase plane belongs to Fig. 5, which is associated with case 2.

Eigenvalues of the closed-loop system as well as the open-loop ones are presented in Fig. 6. For both case 1 and case 2, the averages of the poles are -0.3. The reason is that the state matrices A and A_n in Eq. (6) have the same diagonal elements. Because of the collocation of the actuators and sensors, the product of the input matrix B by gain matrix K by output matrix C has zero elements on its diagonal.

Figure 7 shows the displacements of the masses due to chirp inputs, which increased from 0 to 100 N in 1 s and then reduced zero. Inputs are applied to m_9 and m_{10}, while the actuators and sensors are on $m_{6,7,8}$. A great isolation can be seen on m_{1-5}. The energy that entered the system does not propagate beyond the actuators and is confined to m_{9-10}. $m_{7,8}$ have almost similar amplitude of vibration as open-loop system, but m_6, which carries the inner actuator has shown an isolated behavior. Figure 8 shows the time history of the actuation forces. The maximum actuation force for the inner actuator on m_6 is 19.9 N, for the middle actuator on m_7 is 27.5 N, and on the outer actuator located on m_8 is 49.2 N.

7 Conclusions

Orthogonal eigenstructure control is a numerical output feedback control strategy that can be applied to multi-input multi-output time invariant linear systems. In this paper the actuators and sensors are assumed to be collocated. Similar to numerical eigenstructure assignment methods, singular value decomposition is used to determine the basis of the null space corresponding to each operating eigenvalue. The operating eigenvalues are chosen

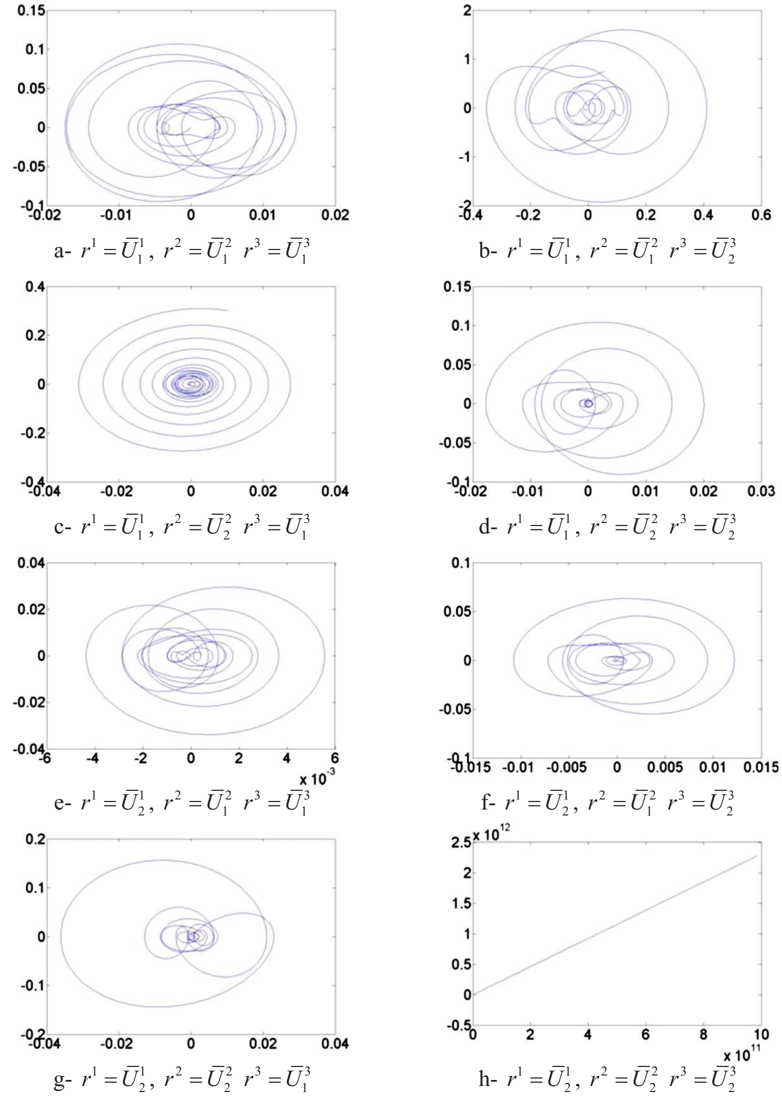

Fig. 5 Phase plane for the closed-loop systems with orthogonal eigenvectors to the open-loop eigenvector

from the open-loop eigenvalues. The algorithm regenerates the open-loop system and simultaneously finds the eigenvectors that are orthogonal to the eigenvectors of the open-loop system. Those open-loop eigenvectors are the intersections of the open-loop eigenvector space and the achievable eigenvector spaces. This method finds those vectors within the achievable eigenvector spaces that are orthogonal to the open-loop ones.

Desired eigenvectors usually are not achievable, which usually leads to some algorithmic error. Moreover, defining the desired

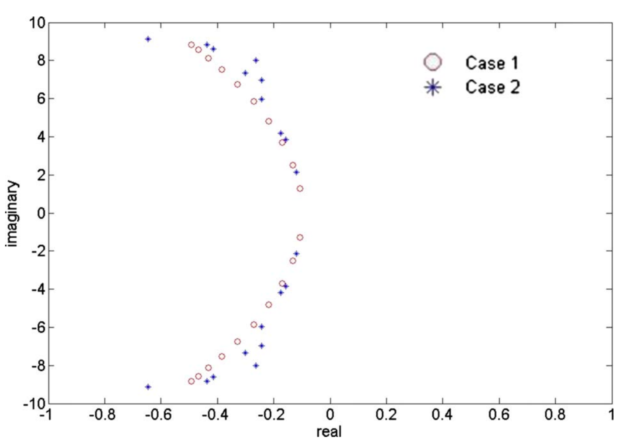

Fig. 6 Eigenvalues of the open-loop system (case 1) and closed-loop system (case 2)

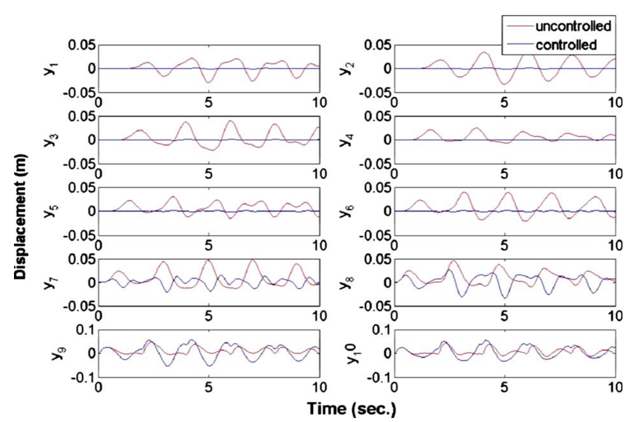

Fig. 7 Displacements of the masses due to a chirp input

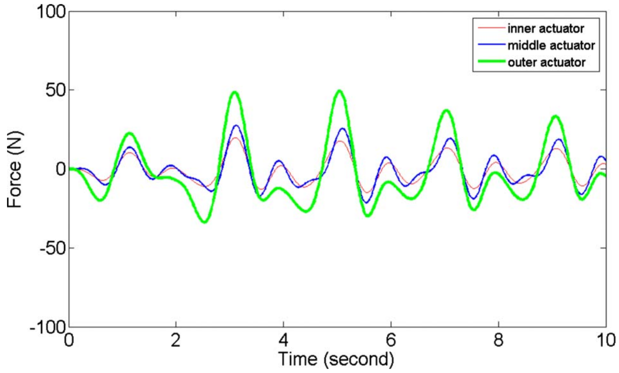

Fig. 8 Actuation forces for the system under a chirp input disturbance

eigenvectors is a challenging task since there is no straightforward method for doing so. This method eliminates the need for defining the desired eigenvectors.

Orthogonal eigenstructure control neither requires the closed-loop eigenvalues to be the same as open-loop ones nor does it specify their locations. As a result, the eigenvalues and corresponding eigenvectors are consistent. Since the actuators and sensors are collocated, the closed-loop and open-loop state matrices have equal traces. Therefore, the average of the closed-loop eigenvalues is the same as open-loop ones.

Nomenclature

A = open-loop state matrix
A_c = closed-loop state matrix
B = input matrix
C = output matrix
E = disturbance input matrix
E_i = modal energy of ith mode
F_i = control input matrix
F_d = disturbance input matrix
f = disturbance
I = identity matrix
K = gain matrix
m = number of inputs (actuators/sensors)
N^i = matrix that spans the null space of ith mode
n = dimension of second order system
r^i = coefficient vector
$S_{\lambda i}$ = augmented matrix associated with λ_i
U_i = left unitary matrix of $S_{\lambda i}$
$\bar{U}^i$ = eigenvalue matrix of $V_{12}^{i*}V_{12}^i$ and $V_{22}^{i*}V_{22}^i$
$\bar{U}_w^i$ = eigenvalue matrix of $V_{22}^{i*}V_{22}^i$, equals to $\bar{U}^i$
$\bar{U}_j^i$ = eigenvalue of $V_{12}^{i*}V_{12}^i$ associated with nonunity eigenvalues
$\bar{U}_J^i$ = eigenvalue of $V_{12}^{i*}V_{12}^i$ associated with unity eigenvalue
V_i = right unitary matrix of $S_{\lambda i}$
V_{12}^i = upper part of N^i
V_{22}^i = lower part of N^i
V = appended matrix of $V_{12}^i r^i$
W = appended matrix of $V_{22}^i r^i$
x = state vector
$\dot{x}$ = time derivative of state vector
ϕ_i = ith closed-loop eigenvalue
ϕ_i^a = achievable eigenvector of ith mode
λ_i = ith operating eigenvalue
$\bar{\lambda}_j^i$ = eigenvalues of $V_{12}^{i*}V_{12}^i$
$\bar{\Lambda}_i$ = eigenvalue matrix of $V_{12}^{i*}V_{12}^i$
$\bar{\Lambda}_w^i$ = eigenvalue matrix of $V_{22}^{i*}V_{22}^i$
Σ_i = matrix of singular values of $S_{\lambda i}$

Appendix A

Matrix A in terms of the physical properties of the system can be written as

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K_s & -M^{-1}C_d \end{bmatrix} \tag{A1}$$

The input matrix can be written as

$$B = \begin{bmatrix} 0 \\ -M^{-1}F_i \end{bmatrix} \tag{A2}$$

The disturbance input matrix can be written as

$$E = \begin{bmatrix} 0 \\ -M^{-1}F_d \end{bmatrix} \tag{A3}$$

Appendix B

Using the orthonormal property, it is seen that

$$\begin{bmatrix} [V_{11}^i]^*_{2n\times 2n} & [V_{21}^i]^*_{m\times 2n} \\ [V_{12}^i]^*_{2n\times m} & [V_{22}^i]^*_{m\times m} \end{bmatrix} \begin{bmatrix} [V_{11}^i]_{2n\times 2n} & [V_{12}^i]_{2n\times m} \\ [V_{21}^i]_{m\times 2n} & [V_{22}^i]_{m\times m} \end{bmatrix} = \begin{bmatrix} I_{2n\times 2n} & 0 \\ 0 & I_{m\times m} \end{bmatrix} \tag{B1}$$

More particularly, it can be seen that the product of the first row block of the $[V_i^*]$ to the second column block of the $[V]_i$ is equal to zero

$$[V_{11}^i]^*_{2n\times 2n}[V_{12}^i]_{2n\times m} + [V_{21}^i]^*_{m\times 2n}[V_{22}^i]_{m\times m} = 0 \tag{B2}$$

It is easy to show that

$$\begin{aligned} S_{\lambda i}\begin{Bmatrix} [V_{12}^i]_{2n\times m} \\ [V_{22}^i]_{m\times m} \end{Bmatrix} &= [A-\lambda_i I|B]\begin{Bmatrix} [V_{12}^i]_{2n\times m} \\ [V_{22}^i]_{m\times m} \end{Bmatrix} \\ &= [U_i][\Sigma_i|0][V_i^*]\begin{Bmatrix} [V_{12}^i]_{2n\times m} \\ [V_{22}^i]_{m\times m} \end{Bmatrix} \\ &= [U_i][\Sigma_i]\{[V_{11}^i]^*_{2n\times 2n} \quad [V_{21}^i]^*_{m\times 2n}\}\begin{Bmatrix} [V_{12}^i]_{2n\times m} \\ [V_{22}^i]_{m\times m} \end{Bmatrix} \\ &= 0 \end{aligned} \tag{B3}$$

The eigenvalue problem can be rewritten as

$$[A-\lambda_i I|B]\begin{Bmatrix} \phi_i \\ KC\phi_i \end{Bmatrix} = [A-\lambda_i I|B]\begin{Bmatrix} V_{12}^i \\ V_{22}^i \end{Bmatrix} r^i = 0, \quad i = 1, \ldots, 2n \tag{B4}$$

where r^i is an arbitrary coefficient vector, which its dimension is equal to m.

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