Elements of matrices *[A], [B], [D], [E], [F]* and *[H]* are defined as follow.

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}, J_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z, z^2, z^3, z^4, z^5, z^6)dz \qquad (i,j = 1,2,3,6) \tag{22}$$

Stress resultants for in-plane components are defined as

$$\begin{Bmatrix} \begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix} \\ \begin{Bmatrix} P_4 \\ P_5 \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [D] & [E] \\ [D] & [E] & [F] & [H] \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} e_4^0 \\ e_5^0 \end{Bmatrix} \\ \begin{Bmatrix} k_4^1 \\ k_5^1 \end{Bmatrix} \end{Bmatrix} \tag{23}$$

where, for example

$$[A] = \begin{bmatrix} A_{44} & A_{45} \\ Sym & A_{55} \end{bmatrix} \tag{24}$$

and

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z, z^2, z^3)dz \qquad (i,j = 4,5) \tag{25}$$

Finally, Using the finite element analysis the equations can be set up in the following form

$$[M]\{\ddot{X}\} + [K]\{X\} = \{F\} \tag{26}$$

4- NATURAL FREQUENCIES FOR DIFFERENT BOUNDARY CONDITIONS

Using quadratic 6 nodes triangular elements to satisfy C^1-continuty of elements, and imposing the boundary conditions, governing equations can be solved to find fundamental frequencies. It is seen that the governing equations are in general dynamic form. To analyze the static behavior of the plate, the stiffness matrix $[K]$ and the mass vector $[M]$ are needed.

For linear problems the local stiffness matrix $[K]$ is independent of element displacement $\{X\}$ and the following relation is valid for all instants of time

$$\{\ddot{X}\} = -\omega^2\{X\} \tag{27}$$

Therefore, the global matrix equations can be written as

$$([K] - \omega^2{}_m[M])\{X\} = \{0\} \tag{28}$$

Natural frequencies of plate can be found from the above relation, where *m* is equal to 7× number of elements that are used.

We assume that the plate is simply supported at two edges while the boundary conditions at the other two edges, are a combination of boundary conditions are applied. These combinations of supports are SS-SS, SS-SC, SS-CC, SS-FC, SS-FS and SS-FF, where S stands for simply support, C for clamped, and F for free boundary conditions. Primary boundary conditions that are used for displacement based finite element analysis are as follow.

For edges located at *x=0* and *x=a* with simply support condition

$$\begin{aligned} v_0(0,y) &= v_0(a,y) = 0 \\ w_0(0,y) &= w_0(a,y) = 0 \\ \phi_y(0,y) &= \phi_y(a,y) = 0 \end{aligned} \tag{29}$$

$$\phi_z(0,y) = \phi_z(a,y) = 0$$

$$\varphi_z(0,y) = \varphi_z(a,y) = 0$$

$$N_1(0,y) = N_1(a,y) = 0$$

$$M_1(0,y) = M_1(a,y) = 0$$

$$P_1(0,y) = P_1(a,y) = 0$$

$$S_1(0,y) = S_1(a,y) = 0$$

For the other two edges (*y=0* and *y=b*), boundary conditions are as follow

For S (simply supported)

$$u_0 = w_0 = \phi_x = \phi_z = \varphi_z = 0 \tag{30}$$

$$N_2 = M_2 = P_2 = S_2 = 0$$

For C (clamped)

$$u_0 = v_0 = w_0 = \phi_x = \phi_y = \phi_z = \varphi_z = 0 \tag{31}$$

$$N_2 = N_6 = N_4 - C_1 N_4 = 0 \quad \text{and} \quad M_2 = M_6 = S_2 = S_6 = P_2 = P_6 = 0$$

5- NUMERICAL SOLUTIONS

Tables 1 through 8 presents nondimensional natural frequencies of square laminated plates with different layers and boundary conditions. Plates are squared with sides (*a=b*).

Natural frequencies of angle plies are presented for a material with the following properties:

Material 1: E_1 = 280 GPa, E_2 = 7 GPa, G_{12}=G_{13}=4.2 GPa, G_{23} =3.5 GPa and $\nu_{12}=\nu_{13}$ =0.25.

Moreover, natural frequencies of cross plies are presented for a material of the following properties:

Material 2: E_1 = 175 GPa, E_2 = 7 GPa, G_{12}=G_{13}=3.5 GPa, G_{23} =1.4 GPa and $\nu_{12}=\nu_{13}$ =0.25.

The following relation is used for presentation of nondimensional fundamental frequencies in this paper.

$$\varpi = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}} \tag{32}$$

Third order shear deformation theory (TRDT) of Reddy has 7 parameters in displacement field and satisfies the vanishing of transverse shear stresses on the boundary planes. Using Reddy displacement field for third order shear deformation theory, a set of dynamic equations for modeling the behavior of a laminated plate is derived. By deriving the dynamic equation of motion and equations in finite element form, natural frequencies of laminated plates with different boundary conditions either cross ply or angle ply is calculated.

Tables 1-4 stands for natural frequencies of angle ply square plates. Table 1 is related to a (45/-45/45/-45/45) angle ply. Table 2 is the results of a (45/-45/45/-45) angle ply, Table 3 shows the natural frequencies of a (45/-45/45) and Table 4 Shows the results for a (45/-45) angle plies. The material properties for these plates are material 1. Tables 5 through 8 are natural frequencies of different cross ply laminates. Table 5 is the results of a six layered (0/90/0/90/0/90) cross ply. Tables 6, 7 and 8 are natural frequencies of a (0/90/0/90), (0/90/0) and (0/90) cross ply respectively. The materials of cross plies are material 2. Table 9 is a comparison between different results of other methods that had been reported.

Looking to the nondimensional natural frequencies of angle ply plates with SS-SS boundary conditions at different length to thickness ratio *a/h*, it is seen that growth of fundamental frequencies with respect to a/h are decreased when *a/h* is increased. Furthermore, with an increase in the number of layers, the rate of increase in natural frequencies will be reduced. The difference between natural frequencies of a two layer angle ply and a three layer angle ply is much more than the difference between a four layer and a five layer plate respectively. By considering the natural frequencies at *a/h*=10, the relative difference between a three layer and a two layer angle ply is 8.84%, between a four layer and a three layer is 5.59%, and between a five layer and a four layer is 0.5%. The same pattern and a close range of relative difference between natural frequencies of plates when other support conditions presents.

Considering nondimensional natural frequencies of SS-SS boundary conditions according to different values of *a/h* shows the same results of angle ply plate; the rate of change in natural frequencies reduces with respect to increase in *a/h*. Taking nondimensional fundamental frequencies at *a/h*=10 shows a relative difference of 8.6% between a three layer and a two layer cross plies. The relative difference is 1.9% between a four layer and a three layer and 0.7% between a six layer and a four layer respectively.

In Table 9, a comparison between natural frequencies of different approach that had been reported in another papers are compare to the results of the present work approach. Table 9 stands for nondimensional natural frequencies of cross plies with different longitudinal and transverse elastic modulus for *a/h=5*.

6- CONCLUSIONS

When comparing the same number of layers, angle plies have higher natural frequencies than cross plies. Increasing the number of layers causes a larger difference between their natural frequencies. It is seen that a cross ply plate with SS-CC boundary conditions, has a higher natural frequency than other types of boundary conditions, but for an angle ply, the SS-SC boundary conditions show a higher natural frequency than other boundary conditions. Also for a cross ply, the natural frequencies of SS-FF and SS-FS boundary conditions are closer to each other and smaller than SS-FC type. On the contrary, natural frequencies for an angle ply, SS-FS and SS-FC boundary conditions are closer to each other and higher than the natural frequency of a plate with SS-FF boundary conditions. Comparison between the natural frequencies of SS-SS plates (a/h=5) as a function of the orthotropicity ratio with the results of several methods reported in [13-15] shows an improvement in the result for this new set of equations.

APPENDIX A: Relationships between strain components and displacement

$$e_1^0 = \frac{\partial u_0}{\partial x} \qquad k_1^0 = \frac{\partial \phi_x}{\partial x} \qquad k_1^1 = -\frac{1}{2}\frac{\partial^2 \phi_z}{\partial x^2} \qquad k_1^2 = -[C_1(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi_x}{\partial x}) + \frac{1}{3}\frac{\partial^2 w_z}{\partial x^2}$$

$$e_2^0 = \frac{\partial v_0}{\partial y} \qquad k_2^0 = \frac{\partial \phi_y}{\partial y} \qquad k_2^1 = -\frac{1}{2}\frac{\partial^2 \phi_z}{\partial y^2} \qquad k_2^2 = -[C_1(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \phi_y}{\partial y}) + \frac{1}{3}\frac{\partial^2 w_z}{\partial y^2}$$

$$e_3^0 = \phi_z \qquad k_3^0 = 2\varphi_z \qquad k_3^1 = 0 \qquad k_3^2 = 0$$

$$e_4^0 = \frac{\partial w_0}{\partial y} + \phi_y \qquad k_4^0 = 0 \qquad k_4^1 = -3C_1(\frac{\partial w_0}{\partial y} + \phi_y) \qquad k_4^2 = 0$$

$$e_5^0 = \frac{\partial w_0}{\partial x} + \phi_x \qquad k_5^0 = 0 \qquad k_5^1 = -3C_1(\frac{\partial w_0}{\partial x} + \phi_x) \qquad k_5^2 = 0$$

$$e_6^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \qquad k_6^0 = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \qquad k_6^1 = -\frac{\partial^2 u_0}{\partial x \partial y}$$

$$k_6^2 = -[C_1(2\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}) + \frac{2}{3}\frac{\partial^2 \varphi_z}{\partial x \partial y}$$

APPENDIX B: Stress-strain relation in different coordinates:

1. constitutive equation of the *n*-th layer where $\overline{\sigma}$ and ē are components of stress and strain in lamina coordinates:

$$\begin{Bmatrix} \overline{\sigma}_1 \\ \overline{\sigma}_2 \\ \overline{\sigma}_3 \\ \overline{\sigma}_4 \\ \overline{\sigma}_5 \\ \overline{\sigma}_6 \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ & \overline{Q}_{22} & \overline{Q}_{23} & 0 & 0 & \overline{Q}_{26} \\ & & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ & & & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ & Sym. & & & \overline{Q}_{55} & 0 \\ & & & & & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \overline{e}_1 \\ \overline{e}_2 \\ \overline{e}_3 \\ \overline{e}_4 \\ \overline{e}_5 \\ \overline{e}_6 \end{Bmatrix} \quad \text{(I)}$$

2. relationships between stresses and strains in both coordinates where σ and e are prescribed in global coordinate.

$$\{\overline{\sigma}\} = [T]\{\sigma\}$$

$$\{\overline{e}\} = [T]\{e\} \quad \text{(II)}$$

3. rotational matrix [*T*] about the transverse direction ***z*** at θ; the angle between the layer coordinates and the global coordinate:

$$[T] = \begin{bmatrix} C^2 & S^2 & 0 & 0 & 0 & 2CS \\ S^2 & C^2 & 0 & 0 & 0 & -2CS \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & -S & 0 \\ 0 & 0 & 0 & S & C & 0 \\ -CS & CS & 0 & 0 & 0 & (C^2 - S^2) \end{bmatrix} \quad \text{(III)}$$

where $C = cos(\theta)$ and $S = sin(\theta)$.

relationship between elastic coefficients at two different coordinate systems:

$$[Q] = [T]^{-1}[\overline{Q}][T] \quad \text{(IV)}$$

where $[\overline{Q}]$ and $[Q]$ are defined in terms of local coordinate of each layer and global coordinate of plate respectively.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ & Q_{22} & Q_{23} & 0 & 0 & Q_{26} \\ & & Q_{33} & 0 & 0 & Q_{36} \\ & & & Q_{44} & Q_{45} & 0 \\ & Sym. & & & Q_{55} & 0 \\ & & & & & Q_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{Bmatrix} \quad \text{(V)}$$

APPENDIX C: Relationships of material properties in rotated coordinate systems

$$Q_{11} = \overline{Q}_{11}C^4 + 2(\overline{Q}_{12} + 2\overline{Q}_{66})C^2S^2 + \overline{Q}_{22}S^4$$

$$Q_{12} = (\overline{Q}_{11} + \overline{Q}_{22} - 4\overline{Q}_{66})C^2S^2 + \overline{Q}_{12}(C^4 + S^4)$$

$$Q_{13} = \overline{Q}_{13}C^2 + \overline{Q}_{23}S^2$$

$$Q_{16} = -\overline{Q}_{22}CS^3 + \overline{Q}_{11}C^3S - CS(C^2 - S^2)(\overline{Q}_{12} + 2\overline{Q}_{66})$$

$$Q_{22} = \overline{Q}_{11}S^4 + 2(\overline{Q}_{12} + 2\overline{Q}_{66})C^2S^2 + \overline{Q}_{22}C^4$$

$$Q_{23} = \overline{Q}_{13}S^2 + \overline{Q}_{23}C^2$$

$$Q_{26} = -\overline{Q}_{22}C^3S + \overline{Q}_{11}CS^3 + CS(C^2 - S^2)(\overline{Q}_{12} + 2\overline{Q}_{66})$$

$$Q_{33} = \overline{Q}_{33}$$

$$Q_{44} = \overline{Q}_{44}C^2 + \overline{Q}_{55}S^2$$

$$Q_{45} = (\overline{Q}_{55} - \overline{Q}_{44})CS$$

$$Q_{55} = \overline{Q}_{55}C^2 + \overline{Q}_{44}S^2$$

$$Q_{66} = (\overline{Q}_{11} + \overline{Q}_{22} - 2\overline{Q}_{12} - 2\overline{Q}_{66})C^2S^2 + \overline{Q}_{66}(C^4 + S^4)$$

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Table 1. Nondimensional natural frequencies of a square angle ply (45/-45/45/-45/45) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	11.196	19.059	21.371	24.302	25.949
ss-sc	11.752	20.501	27.466	29.465	30.830
ss-cc	11.595	19.333	26.303	29.581	30.018
ss-ff	4.413	6.123	7.161	7.873	8.163
ss-fs	6.953	9.593	11.790	12.480	13.001
ss-fc	6.988	9.788	11.847	13.477	13.995

Table 2. Nondimensional natural frequencies of a square angle ply (45/-45/45/-45) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	11.151	18.964	21.296	24.133	25.897
ss-sc	11.705	20.399	27.370	29.260	30.768
ss-cc	11.549	19.237	26.211	29.375	29.958
ss-ff	4.395	6.093	7.136	7.818	8.147
ss-fs	6.925	9.545	11.749	12.393	12.975
ss-fc	6.960	9.739	11.806	13.383	13.967

Table 3. Nondimensional natural frequencies of a square angle ply (45/-45/45) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	10.560	17.959	20.167	22.854	24.524
ss-sc	11.085	19.318	25.919	27.709	29.137
ss-cc	10.937	18.217	24.822	27.818	28.370
ss-ff	4.162	5.770	6.758	7.404	7.715
ss-fs	6.558	9.039	11.126	11.736	12.287
ss-fc	6.591	9.223	11.180	12.674	13.227

Table 4. Nondimensional natural frequencies of a square angle ply (45/-45) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	9.701	16.499	18.528	20.996	22.530
ss-sc	10.183	17.747	23.812	25.456	26.768
ss-cc	10.048	16.736	22.804	25.556	26.063
ss-ff	3.824	5.301	6.208	6.802	7.088
ss-fs	6.025	8.304	10.222	10.782	11.288
ss-fc	6.055	8.473	10.271	11.643	12.151

Table 5. Nondimensional natural frequencies of a square cross ply (0/90/0/90/0/90) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	9.016	12.300	14.130	15.040	15.177
ss-sc	9.256	14.974	17.343	18.781	18.960
ss-cc	10.287	17.522	23.448	30.392	34.179
ss-ff	3.450	3.782	3.828	3.909	3.936
ss-fs	3.818	4.283	4.406	4.461	4.617
ss-fc	5.189	6.243	6.516	6.809	6.830

Table 6. Nondimensional natural frequencies of a square cross ply (0/90/0/90) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	8.953	12.214	14.032	14.936	15.071
ss-sc	9.191	14.870	17.222	18.650	18.829
ss-cc	10.215	17.400	23.285	30.180	33.942
ss-ff	3.426	3.755	3.802	3.882	3.909
ss-fs	3.792	4.253	4.376	4.430	4.585
ss-fc	5.153	6.199	6.471	6.761	6.783

Table 7. Nondimensional natural frequencies of a square cross ply (0/90/0) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	8.935	12.19	14.004	14.906	15.041
ss-sc	9.173	14.84	17.188	18.613	18.791
ss-cc	10.195	17.365	23.239	30.12	33.874
ss-ff	3.419	3.748	3.794	3.874	3.901
ss-fs	3.784	4.245	4.367	4.421	4.576
ss-fc	5.143	6.187	6.458	6.748	6.769

Table 8. Nondimensional natural frequencies of a square cross ply (0/90) laminated composite plate with different support conditions.

a/h	5	10	20	50	100
ss-ss	8.220	11.215	12.884	13.714	13.838
ss-sc	8.439	13.653	15.813	17.124	17.288
ss-cc	9.379	15.976	21.380	27.710	31.164
ss-ff	3.145	3.448	3.490	3.564	3.589
ss-fs	3.481	3.905	4.018	4.067	4.210
ss-fc	4.732	5.692	5.941	6.208	6.227

Table 9. Nondimensional natural frequencies of a simply supported square laminated composite cross ply (0/90/90/0) with different E_1/E_2 ratios and *a/h*=5.

E_1/E_2	Elasticity[13][*]	HSDT[†]	HSDT[14][‡]	HSDT[15][§]	FSDT[15]	CLPT[15]
10	8.2103	8.2681	8.2718	8.294	8.2982	10.65
20	9.5603	9.5112	9.5263	9.5439	9.5671	13.948
30	10.272	10.2601	10.272	10.284	10.326	16.605
40	10.752	10.7634	10.787	10.794	10.854	18.891

[*] Results obtained by applying finite difference method to the Three dimensional elasticity equations
[†] Present work, FEM of the equations governing in this paper
[‡] Results obtained by Levy type solution with a different HSDT
[§] Results obtained by using other HSDT, FSDT and CLPT with Levy type solution